List colouring of graphs and hypergraphs

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Vertex colouring

A *vertex colouring* of a graph G is a map

$$c : V(G) \to \mathbb{N} \quad \text{such that} \quad c(u) \neq c(v) \text{ whenever } uv \text{ is an edge}$$

Here $\mathbb{N}$ is the palette of available colours.

The *chromatic number* of G is

$$\chi(G) = \min\{k : \text{ there is a colouring } c : V(G) \to \{1, \ldots, k\}\}$$

List colouring

Suppose now we assign a *list* of colours to each vertex, ie

$$L : V(G) \to \mathcal{P}(\mathbb{N})$$

We say G is *L-choosable* if there is a colouring

$$c : V(G) \to \mathbb{N} \quad \text{with} \quad c(v) \in L(v) \text{ for all v}$$

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In ordinary colouring, the lists are the same for every v

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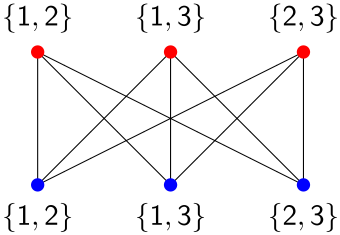
Clearly $\chi(G) \leq \chi_\ell(G)$

χ_ℓ can be bigger than χ

$K_{3,3}$ not 2-choosable: $\chi = 2$, $\chi_\ell \geq 3$



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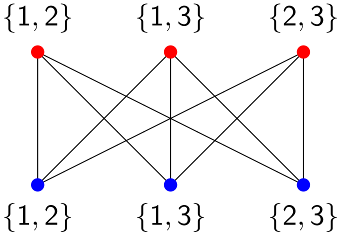
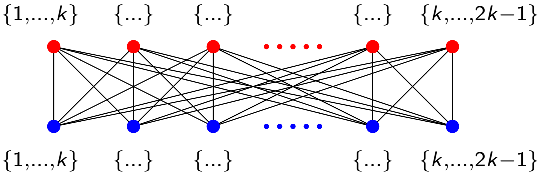


More generally, $K_{m,m}$ is not k-choosable if $m \geq \binom{2k-1}{k}$



Property B

$\mathcal{F} \subset \mathcal{P}(X)$ has *Property B* if the hypergraph $(X, \mathcal{F})$ is bipartite ie

$$\exists\, S \subset X \;:\; \forall F \in \mathcal{F} \quad \emptyset \neq S \cap F \neq F$$

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$m(k)$= $\min\{\, |\mathcal{F}| \ : \forall F \in \mathcal{F}\ |F| = k,\ \mathcal{F}$ does *not* have property $B\}$

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Theorem (Erdős-Rubin-Taylor)

$$m(k) \;\leq\; \min\{|G| : \chi(G) = 2,\, \chi_\ell(G) > k\} \;\leq\; 2m(k)$$

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Proof: (lower bound)

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Proof: (lower bound)

let $G = K_{a,b}$, $a + b < m(k)$, $L : V(G) \to \mathcal{P}(\mathbb{N})$, $|L(v)| = k$.

Let $\mathcal{F} = \{L(v) : v \in G\}$. Then $\mathcal{F}$ has property B.

Hence $\exists S$ such that $\forall v \ \emptyset \ne S \cap L(v) \ne L(v)$. Likewise $\overline{S} = \mathbf{N} - S$.

Colour one side of G with colours from S, other side from $\overline{S}$.

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Proof: (upper bound)

Let $|\mathcal{F}| = m(k)$, $\mathcal{F}$ does not have property B.

Let $G = K_{m(k),m(k)}$. Let L assign lists from $\mathcal{F}$ to each side of G.

If G is L-choosable, let S be colours used on one side, $\overline{S}$ other side, and $\forall F \in \mathcal{F}$ both $\emptyset \neq S \cap F$ and $\emptyset \neq \overline{S} \cap F$, so $\mathcal{F}$ has property B.

χ_ℓ and average degree

Clearly $m(k) \leq \binom{2k-1}{k} < 4^k$.

Hence $\exists$ G with $\chi_\ell(G) > \frac{1}{2}\log_2 d \quad (d = \text{average degree of } G)$

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Erdős: $\quad 2^{k-1} \leq m(k) \leq k^2 2^{k+1}$

Hence $\exists$ G with $\chi_\ell(G) = (1 + o(1))\log_2 d$

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Theorem (Alon '00)

Every G of average degree d satisfies $\chi_\ell(G) \ge (\frac{1}{2} + o(1)) \log_2 d$.

(This is tight to within a factor of 2.)

χ_ℓ and average degree for hypergraphs

$G = (V, E)$ is an *r-uniform hypergraph* if $e \in E \Rightarrow e \subset V$, $|e| = r$

$c : V(G) \to \mathbb{N}$ is a colouring if no edge e is monochromatic

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We require G to be *simple*: i.e., $|e \cap f| \leq 1$ for distinct edges e, f

χ_ℓ for simple hypergraphs

There are easy examples of simple G with $\chi_\ell(G) = O(\log_r d)$

$r = 3$

- Haxell+Pei '09: $\chi_\ell(G) = \Omega(\frac{\log d}{\log \log d})$ for Steiner systems
- Haxell+Verstraëte '10: $\chi_\ell(G) = \Omega(\sqrt{\frac{\log d}{\log \log d}})$ for regular

general r

- Alon+Kostochka '10: $\chi_\ell(G) = \Omega((\log d)^{\frac{1}{r-1}})$

Theorem (Saxton+T '12)

If G is r-uniform, simple, d-regular then $\chi_\ell(G) = \Omega(\log d)$.

χ_ℓ and average degree

Theorem (Saxton+T '13+)

If G is r-unif, simple, average degree d then $\chi_\ell(G) \geq \frac{1}{(r-1)^2} \log_r d$

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Remark: method handles non-simple too
Remark: closes gap of $1/2$ for graphs ($r = 2$)

A sketch of a proof of something

Let $\mathcal{I}$ be all the independent sets in G
Let $\max\{|I| : I \in \mathcal{I}\} \leq (1-\gamma)n$ (eg $\gamma = 1/r$ for regular G)

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L assigns lists size k randomly from palette of t colours
G L-choosable $\Rightarrow \exists\ I_1, \ldots, I_t \in \mathcal{I} : \forall v,\ v \in I_j$ some $j \in L(v)$
we say L *fits* $I_1, \ldots, I_t$

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"on average" $v \in I_j$ for $\leq (1-\gamma)t$ sets I_j
so $\Pr(L \text{ fits } I_1, \ldots, I_t) \leq \prod_v (1-\gamma^k) \leq e^{-n\gamma^k}$

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If $|\mathcal{I}| \leq e^{n/d}$ then $\Pr(L \text{ fits some } I_1, \ldots, I_t) \leq e^{nt/d} e^{-n\gamma^k} = o(1)$
provided $k < c \log d$ — that is, $\chi_\ell(G) \geq c \log d$

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Replace $\mathcal{I}$ and $I_1, \ldots, I_t \in \mathcal{I}$ by $\mathcal{C}$ and $C_1, \ldots, C_t \in \mathcal{C}$ — *containers*
where $|\mathcal{C}| \leq e^{n/d}$, $\forall C \in \mathcal{C}\ |C| \leq (1-\gamma)n$, $\forall I \in \mathcal{I}\ \exists C \in \mathcal{C}\ I \subset C$

Bullet points

Want $|\mathcal{C}| \le e^{n/d}$, $\forall C \in \mathcal{C}\ |C| \le (1-\gamma)n$, $\forall I \in \mathcal{I}\ \exists C \in \mathcal{C}\ I \subset C$

- Sapozhenko did it for regular graphs (Cameron-Erdős)
- simple regular hypergraphs
- $K_{d,n-d}$
- degree measure: for $S \subset V$, $\mu(S) = \frac{1}{nd}\sum_{v \in S} d(v)$
  then $\mu(I) \le 1 - 1/r$; we get $\mu(C) \le 1 - 1/r$!
- simple hypergraphs
- all hypergraphs
- other simple expectation arguments

Sapozhenko's method

Let G be a d-regular graph. $V(G) = \{1, \ldots, n\}$.
Let $\epsilon > 0$.

INPUT `independent set` I
put $T = \emptyset$
for $v = 1, ., n$: if $v \in I$ and $|\Gamma(T \cup \{v\})| \geq |\Gamma(T)| + \epsilon d$, add v to T

Afterwards, observe $T \subset I$ and $|T| \leq n/\epsilon d$.

INPUT `set` $T \subset V(G)$
put $C = V(G) - \Gamma(T)$
for $v = 1, ., n$: if $v \notin T$ but $|\Gamma(T \cup \{v\})| \geq |\Gamma(T)| + \epsilon d$, take v from C

Afterwards, note if T came from first algorithm then $I \subset C$.
Also, $\Delta(G[C]) \leq \epsilon d$ and G is d-regular so $|C| \leq n/(2 - \epsilon)$.

An algorithm (or two)

INPUT an r-graph G on vertex set $[n]$
an $(s+1)$-multigraph P_{s+1} on vertex set $[n]$
parameters $\tau, \zeta > 0$
a subset $I \subset [n]$
a subset $T_s \subset [n]$

OUTPUT an s-multigraph P_s on vertex set $[n]$
a subset $T_s \subset [n]$
a subset $C_s \subset [n]$

put $E(P_s) = \emptyset$ and $\Gamma_s = \emptyset$
put $T_s = \emptyset$
put $C_s = [n]$

for $v = 1, 2, \ldots, n$ do:
  let $F = \{f \in [v+1, n]^{(s)} : \{v\} \cup f \in E(P_{s+1}),$ and $\forall \sigma \in \Gamma_s\ \sigma \not\subset f\}$
  if $|F| \geq \zeta \tau^{r-s-1} d(v)$ and $v \in I$, add v to T_s
  if $|F| \geq \zeta \tau^{r-s-1} d(v)$, remove v from C_s
  if $v \in T_s$ then
    add F to $E(P_s)$
    for each $u \in [v+1, n]$, if $d_s(u) > \tau^{r-s} d(u)$, add $\{u\}$ to Γ_s
    for each $\sigma \in [v+1, n]^{(>1)}$, if $d_s(\sigma) > 2^s \tau d_{s+1}(\sigma)$, add σ to Γ_s