On the core of an integer

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1. Definition and motivation

$k(n) := \prod_{p|n} p$: core or squarefree kernel of n.

Arises in sieve problems, Artin conjecture, Dirichlet divisor problem, simultaneous representation of primes by quadratic forms, size of coefficients of modular forms, mathematical logic (Erdős–Woods problem), Siegel zero for L-functions, *abc* conjecture, etc...

Erdős: $\Psi_m(x) := \sum_{n \leqslant x,\, k(n) \mid m} 1.$

Evaluate $K_1(x) := \sum_{m \leqslant x} \Psi_m(x).$

$$K_1(x) = \sum_{n \leqslant x} \sum_{\substack{m \leqslant x \\ k(n) \mid m}} 1 = \sum_{n \leqslant x} \left\lfloor \frac{x}{k(n)} \right\rfloor$$

$$= xK(x) + O(x), \qquad K(x) := \sum_{n \leqslant x} \frac{1}{k(n)}$$

Erdős: $K(x) = \mathrm{e}^{\{1+o(1)\}\sqrt{8 \log x / \log_2 x}}$?

De Bruijn (1962): yes!

Erdős: $K_2(x) := \sum_{n \leqslant x} \frac{n}{k(n)} = o\Big(\sum_{n \leqslant x} \frac{x}{k(n)}\Big)$?

De Bruijn & van Lindt (1963): yes!

What about genuine asymptotic formulae for

$$K(x) := \sum_{n \leqslant x} \frac{1}{k(n)},$$

$$K_1(x) := \sum_{n \leqslant x} \left\lfloor \frac{x}{k(n)} \right\rfloor,$$

$$K_2(x) := \sum_{n \leqslant x} \frac{n}{k(n)}?$$

2. Characteristics of the problem

$N(x,y) := \sum_{\substack{n\leqslant x\\ k(n)\leqslant y}} 1$??

Resembles

$$\Psi(x,y) := \sum_{\substack{n\leqslant x\\ P^+(n)\leqslant y}} 1$$

$$= \frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} \prod_{p\leqslant y}\Big(1-\frac{1}{p^s}\Big)^{-1}\frac{x^s}{s}\,\mathrm{d}s \quad ?$$

Not so much:

$$N(x,y)=\frac{1}{(2\pi i)^2}\int_{\sigma+i\mathbb{R}}\int_{\kappa+i\mathbb{R}}F(s,z)\frac{x^s y^z}{sz}\,\mathrm{d}s\,\mathrm{d}z$$

$$F(s,z):=\sum_{n\geqslant 1}\frac{1}{n^s k(n)^z}=\prod_p\Big(1+\frac{1}{p^z(p^s-1)}\Big)$$

3. First attempt

Erdős–Wintner: $\mathrm{dens}\{n:k(n)/n\leqslant \mathrm{e}^{-t}\}=a(t)$,

$$\widehat{a}(\vartheta)=\frac{6}{\pi^2}\prod_p\Big(1+\frac{1}{(p+1)(p^{1+i\vartheta}-1)}\Big)$$

$$a(t)=\frac{6}{\pi^2}\sum_{m\leqslant \mathrm{e}^t}\frac{1}{m\psi(m)},\ \big(\psi(m):=\textstyle\prod_{p|m}(p+1)\big).$$

Thus we expect

$N(x,y) \approx yF(v)$, with $v = \log(x/y)$, and

$$F(t) := 1 + \int_0^t \mathrm{e}^u\{1 - a(u)\}\,\mathrm{d}u \qquad (t \geqslant 0).$$
$$= \frac{6}{\pi^2} \sum_{m \geqslant 1} \frac{\min(1, \mathrm{e}^t/m)}{\psi(m)}$$

Joint work with O. Robert on $N(x,y)$ (120 pp).

Firts observation: this may be proved analytically or using the canonical representation $n = kms$, with $k = k(m)$, $\mu(s)^2 = 1$, $(s,k) = 1$.

Let $\mathcal{Y}_x := \mathrm{e}^{\frac{1}{4}\sqrt{2\log x}(\log_2 x)^{3/2}}$.

Then $N(x, y) \sim yF(v)$ when $y > \mathcal{Y}_x^c,\ c > \frac{3}{2}$.

Proof elementary provided we know the local behavior of F and ratio to its Rankin upper bound.

Saddle-point method:

$$F(v) = \frac{1}{2\pi i} \int_{\sigma + i\mathbb{R}} \mathrm{e}^{vs} \prod_p \left(1 + \frac{1 - p^{1-s}}{p(p^s - 1)}\right) \frac{\mathrm{d}s}{s(1-s)}$$

As $v \to \infty$,

$$F(v) \sim \frac{\mathrm{e}^{v\sigma_v + g(\sigma_v)}}{\sigma_v \sqrt{2\pi g''(\sigma_v)}} \sim \frac{\mathrm{e}^{v\sigma_v + g(\sigma_v)}}{2\sqrt{\pi}(2v/\log v)^{1/4}}$$

where $g(\sigma) := \sum_p \log\left(1 + \frac{1 - p^{\sigma - 1}}{p(p^\sigma - 1)}\right)$,

$g'(\sigma_v) + v = 0.$

$$\sigma_v \approx \sqrt{\frac{2}{v\log v}}\left\{1+\sum_{k\geqslant 0}\frac{P_k(\log_2 v)}{(\log v)^k}\right\},$$

$$\log F(v) \approx \sqrt{\frac{8v}{\log v}}\left\{1+\sum_{j\geqslant 1}\frac{Q_j(\log_2 v)}{(\log v)^j}\right\}$$

$P_k,\ Q_j \in \mathbb{R}[X]$, $\deg P_k \leqslant k$, $\deg Q_j \leqslant j$.

Questions:

1. How far is $N(x,y) \sim yF(v)$ valid?

2. What does $N(x,y)$ look like when the above asymptotics fail?

4. Saddle-point estimates

$$\mathscr{F}(s,z) := \prod_p \left(1 + \frac{1}{p^z(p^s-1)}\right),$$

$$N(x,y) := \frac{1}{(2\pi i)^2} \iint \mathscr{F}(s,z) \frac{x^s y^z}{sz}\,\mathrm{d}s\,\mathrm{d}z$$

Rankin: $N(x,y) \leqslant x^\alpha y^\beta \mathscr{F}(\alpha,\beta)$.

Double saddle-point:

$$N(x,y) \sim \frac{x^\alpha y^\beta \mathscr{F}(\alpha,\beta)\Phi(\lambda/\alpha)}{2\pi\alpha\beta\sqrt{\delta(\alpha,\beta)}} \quad \left(y > \mathrm{e}^{(\log_2 x)^{3+\varepsilon}}\right)$$

$$N(x,y) \sim \frac{x^\alpha y^\beta \mathscr{F}(\alpha,\beta)\Phi(\lambda/\alpha)}{2\pi\alpha\beta\sqrt{\delta(\alpha,\beta)}}$$

$\lambda := \alpha + \beta - 1$, $\delta :=$ Hessian associated to $\log \mathscr{F}$,

Φ elementary...

$$\Phi(u) := \sqrt{\frac{2\pi D'(u)}{\Gamma(u)}}\mathrm{e}^{uD(u)-\exp D(u)}, \quad D(u) := \frac{\Gamma'(u)}{\Gamma(u)}.$$

Unusual behavior:

$\Phi(\lambda/\alpha) \to 1 \Leftrightarrow \lambda/\alpha \to 0$ (small y),

but $\Phi(\lambda/\alpha) \to \sqrt{2\pi}/\mathrm{e} \Leftrightarrow \lambda/\alpha \to \infty$ (large y).

Explanation: for large y, the double integral is *not* dominated by a small neighborhood of the saddle-point: we need the *indirect saddle-point method.*

5. Consequences

5·1. Explicit formulae

$$\mathcal{Y}_x := \mathrm{e}^{\frac{1}{4}\sqrt{2\log x}(\log_2 x)^{3/2}}, \; M_x := \sqrt{2\log x \log_2 x}\,\log_3 x.$$

Then:

(i) $N(x,y) \sim yF(v) \Leftrightarrow y > \mathcal{Y}_x \mathrm{e}^{-3M_x/8} \mathrm{e}^{\psi_x \sqrt{\log x \log_2 x}}$, $\psi_x \to \infty$.

(ii) $\Delta := \mathrm{e}^{-\gamma} v\sigma_v / y^{\sigma_v}$.
Then $N(x,y) \sim yF(v)\mathrm{e}^{-\Delta}$ for $y > \mathcal{Y}_x \mathrm{e}^{-3M_x/4}$.

(iii) $N(x,y) \sim yF(v)\mathrm{e}^{-D(\lambda/\alpha)}$ for $\sqrt{\mathcal{Y}_x}\mathrm{e}^{15M_x/8} < y \leqslant x$.

(iv) $\forall c \in]0,1[\ \exists b = b_c$:
$N(x,y) = yF(v)^{c+o(1)} \Leftrightarrow y = \mathrm{e}^{\{b+o(1)\}\sqrt{\log x \log_2 x}}$
[b : solution to an explicit equation depending on c.]

(v) $N(x,y) = yF(v)^{o(1)} \Leftrightarrow (\log y)/\sqrt{\log x \log_2 x} \to 0$.

(vi) For some explicit $\mathscr{K} = \mathscr{K}(x,y) \asymp \Delta$, we have
$N(x,y) \sim yF(v)\mathrm{e}^{-\{1+o(1)\}\mathscr{K}} \quad (x \geqslant y \geqslant 2)$.

5·2. *Lost factor in Rankin's bound*

Recall $\Delta := \mathrm{e}^{-\gamma} v\sigma_v / y^{\sigma_v}$. Let $\mathrm{v} := v + 2$.

When $\log y > (\log_2 x)^{3+\varepsilon}$, we have

$$\mathscr{R} := \frac{x^\alpha y^\beta \mathscr{F}(\alpha, \beta)}{N(x,y)}$$

$$\asymp \begin{cases} \dfrac{\log(2 + 1/\Delta)}{\sqrt{1+\Delta}} \mathrm{v}^{3/4} (\log \mathrm{v})^{1/4} & \text{if } \dfrac{\log y}{\sqrt{\log x \log_2 x}} > 1, \\ \sqrt{\dfrac{\log_2 x}{\log_2 y} \log y} & \text{if } \dfrac{\log y}{\sqrt{\log x \log_2 x}} \leqslant 1. \end{cases}$$

In particular, $\mathscr{R} \ll (\log x)^{5/4}$.

5·3. Local behavior (semi-asymptotic formulae)

Slow variation of α and β as functions of x, y yield asymptotics for $N(tx,y)/N(x,y)$, $N(x,ty)/N(x,y)$ for moderate t.

For instance, $\forall b > 1$,

$$N(2x,y) \sim N(x,y) \Leftrightarrow y = o(x)$$

$$N(x,2y) \sim 2N(x,y) \Leftrightarrow \log y > (\log x)^{1/2+o(1)}$$

$$N(x,2y) \sim 2^b N(x,y) \Leftrightarrow \log y = (\log x)^{1/(b+1)+o(1)}.$$

5·4. De Bruijn's functions

$$K(x) := \sum_{n\leqslant x} \frac{1}{k(n)} = \int_1^\infty \frac{N(x,y)}{y^2}\,\mathrm{d}y.$$

Study of $N(x,y) \Rightarrow$ some neighborhood of $y = \mathcal{Y}_x$ dominates.

Local behavior $\Rightarrow$ expansion around $y = \mathcal{Y}_x$.

$$K(x) \approx \tfrac{1}{2}\mathrm{e}^\gamma F(\log x)(\log_2 x)\Big\{1 + \sum_{j\geqslant 1} \frac{R_j(\log_3 x)}{(\log_2 x)^j}\Big\}.$$

$R_1(z) = -z$, $R_2(z) = 2z - \frac{1}{3}\pi^2$.

Same technique yields optimal bound, and indeed asymptotic formula, in Erdős' conjecture:

$$K_2(x) = K_1(x)\sigma_{\log x}\left\{1 + O\left(\sqrt{\frac{\log_2 x}{\log x}}\right)\right\}$$

$$\sim \frac{\sqrt{2}K_1(x)}{\sqrt{\log x \log_2 x}} \qquad (x \to \infty).$$

Recall $K_1(x) \sim xK(x) = \sum_{n \leqslant x} x/k(n)$,

$$K_2(x) := \sum_{n \leqslant x} n/k(n).$$

5·5. abc conjecture

Joint work with O. Robert and C. Stewart.

Masser–Oesterlé :

$a + b = c$, $(a, b) = 1 \Rightarrow c \ll_\varepsilon k(abc)^{1+\varepsilon}$.

Deep consequences: Fermat, Siegel zero of quadratic L-functions, Szpiro's conjecture, etc.

Using the sole hypothesis that $k(c)$ behaves independently of $k(a)$ and $k(b)$, we can show that results on $N(x, y)$ imply

$$c \ll kF(\tfrac{2}{3}\log k)^{3 - B_0/\log_2 k},$$

$$c > kF(\tfrac{2}{3}\log k)^{3 - B_1/\log_2 k} \quad \text{(i.o.)}.$$

More precise:

$\exists \mathcal{H}(k)$, explicitely defined,

$\mathcal{H}(k) = F(\tfrac{2}{3}\log k)^{\{\log 4 + o(1)\}/\log_2 k}$, such that

$$c \ll \frac{kF(\tfrac{2}{3}\log k)^3(\log k)^3}{\mathcal{H}(k)},$$

$$c > \frac{kF(\tfrac{2}{3}\log k)^3}{\mathcal{H}(k)(\log k)^2} \quad \text{(i.o.)}.$$

6. Methods

Aside of classical tools (Rankin-type bounds, saddle-point method, contour integration, convolution), we need some less usual devices.

6·1. Estimates of sums over primes with two parameters

Need complete uniformity.

For instance (recall $D := \Gamma'/\Gamma$)

$$\sum_p \frac{(\log p)^2 p^\kappa (p^\sigma - 1)}{\{1 + p^\kappa (p^\sigma - 1)\}^2}$$

$$= \frac{1}{\sigma^2} D'\Big(1 + \frac{\kappa - 1}{\sigma}\Big) + O\Big(\frac{\log(1/\sigma)}{\sigma}\Big)$$

uniformly for $0 < \sigma < \frac{1}{4}$, $\kappa + \sigma > 1$, $2^\kappa(2^\sigma - 1) < 1$.

Note that main term has size $\asymp 1/\sigma^2$ for $\kappa \ll 1$.

Also need short interval estimates (Montgomery–Vaughan, Heath-Brown), and upper bounds for exponential sums over primes.

For instance, with $W(z) := \exp z^{1/4}$, we have for $z \geqslant 3$, $1 < \sigma < W(\log z)^{c_1}$, $|t| \leqslant \exp\{(\log z)^{37/36}\}$,

$$\sum_{p>z} \frac{\log p}{p^{\sigma+it}} = \frac{z^{1-\sigma-it}}{\sigma - 1 + it} + O\Big(\frac{z^{1-\sigma}}{W(\log z)^{c_2}}\Big).$$

6·2. Indirect saddle point method

Corresponds to the case when a small neighborhood of the saddle-point is not sufficient.

We then appeal to Laplace inversion.

This is necessary for large values of y.

Example: for suitable small $\mathfrak{u}$, $\mathfrak{v}$,

$$\mathscr{D} := [\alpha - i\mathfrak{u}, \alpha + i\mathfrak{u}] \times [\beta - i\mathfrak{v}, \beta + i\mathfrak{v}]$$

$$\int_{\mathscr{D}} \frac{\mathscr{F}(s,z)x^s y^z}{(2i\pi)^2 sz}\,\mathrm{d}s\,\mathrm{d}z \approx \frac{y}{2i\pi}\int_{\alpha-i\mathfrak{u}}^{\alpha+i\mathfrak{u}} \frac{\mathrm{e}^{sv}\mathscr{G}(s)\psi(s)}{s}\,\mathrm{d}s$$

with $\mathscr{G}(s) := \mathrm{Res}(\mathscr{F}(s,z); z = 1-s)$.

In the left side, relevant values of $\mathfrak{v}$ turn out to be too large to apply the saddle-point method and restricting it spoils the error.

The z-integral is evaluated by estimates of the type

$$\int_{-T}^{T} \Gamma\Big(\frac{\lambda + i\tau + it}{s}\Big)\mathrm{e}^{t\zeta}\,\mathrm{d}t \approx 2\pi s \mathrm{e}^{-\exp(is\zeta)+i(\lambda+i\tau)\zeta}$$

for suitable values of $\Re e\, s$, T and τ.

6·3. *Asymptotic analysis*

Rouché's method for Lagrange's theorem.

Let $f(z)$ be holomorphic in a neighborhood of $z = 0$ and $f(0) \neq 0$, then, for small $|w|$ and r, $wf(z) = z$ has a unique solution $z = z(w)$ in $|z| \leqslant r$.

Moreover $z(w) = \sum_{n\geqslant 1} a_n w^n$,

with $a_n := \dfrac{1}{n!}\left[\dfrac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} f(z)^n\right]_{z=0} \quad (n \geqslant 1)$.

Sketch proof.

By Rouché $z - wf(z)$ has exactly one zero such that $|z| \leqslant r$ and Cauchy's formula yields

$$z(w) = \frac{1}{2\pi i} \oint_{|z|=r} \frac{1 - wf'(z)}{z - wf(z)} z \, \mathrm{d}z$$

$$= \frac{1}{2\pi i} \oint_{|z|=r} \{1 - wf'(z)\} \sum_{n \geqslant 0} \frac{w^n f(z)^n}{z^n} \, \mathrm{d}z.$$

Asymptotic expansion for certain integrals.

Consider $\displaystyle\int_{\log 2}^{\infty} \frac{\mathrm{e}^{u}}{(\mathrm{e}^{\sigma u}-1)\{1+\mathrm{e}^{u}(1-\mathrm{e}^{-\sigma u})\}}\,\mathrm{d}u$

- Define X such that $\mathrm{e}^{X}(1-\mathrm{e}^{-\sigma X})=1$,
- Expand the integrand into series of $\mathrm{e}^{u}(1-\mathrm{e}^{-\sigma u})$ or $1/\{\mathrm{e}^{u}(1-\mathrm{e}^{-\sigma u})\}$ according to whether $u \leqslant X$ or $u > X$ (semi-convergent series);
- Expand into negative powers of X;

- Expand X into series of $\log_2 1/\sigma$ and $1/\log(1/\sigma)$;
- Invert summations (non-trivial).