

# Local chromatic number of graphs and digraphs

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a survey

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# Definition

Erdős, Füredi, Hajnal, Komjáth, Rödl, Seress  
1986

$$\psi(G) := \min_c \max_v |\{c(u) : \{u,v\} \in E(G)\}| + 1$$

The minimization is over all proper colorings  $c$ .

$\psi(G)$  is the minimum number of colors that must appear in the most colorful closed neighborhood of a vertex in any proper coloring.

Obviously:  $\psi(G) \leq \chi(G)$ .

always = ?

Thm. (EFHKRS 1986):  $\forall k, \exists G : \psi(G) = 3, \chi(G) > k$ .

revisiting local chromatic number

Körner, Pilotto, Simonyi, 2005

+ directed version

related to “Sperner capacity” in information theory

(KPS) + an information transmission problem

(Shanmugam-Dimakis-Langberg 2013)

+ study of  $\psi$  and  $\psi_d$  as graph parameters, e.g.:

Thm. (KPS 2005):  $\chi^*(G) = \psi^*(G) \leq \psi(G)$

$\chi^*$  = fractional chromatic number

Thm. (KPS 2005):  $\chi^*(G) \leq \psi(G) \leq \chi(G)$

$\chi^*(G) = \chi(G) \quad \Rightarrow \quad \psi(G)$  is determined

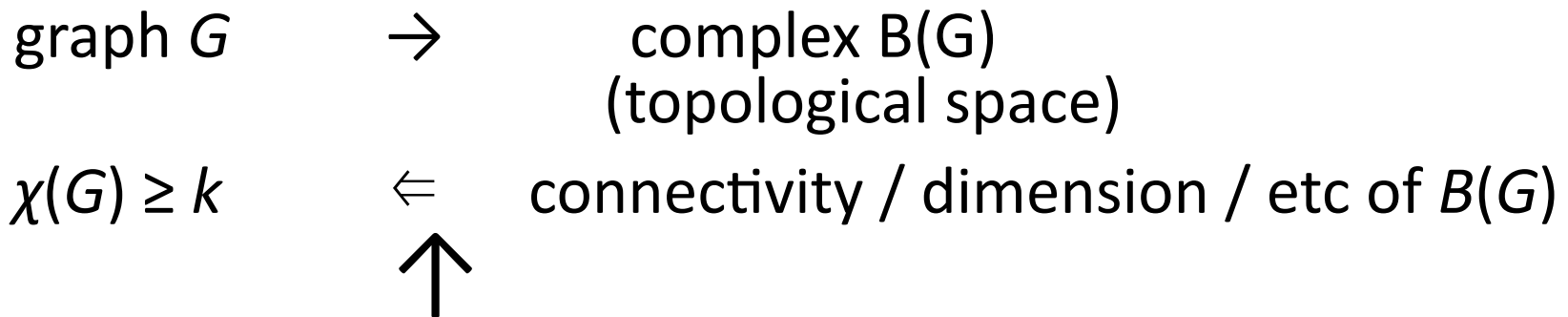
examples for  $\chi^*(G) \ll \chi(G)$

- Kneser graph
- Schrijver graphs
- (generalized) Mycielski graphs
- shift graphs
- ????

$\chi(G)$  is found through  
“topological methods”

# topological method

started by Lovász, 1978



often using the Borsuk-Ulam theorem

Thm. (Lusternik-Schnirelmann, 1934  $\approx$  B-U)

$S^{n-2}$  covered by open antipodal-free sets  $\Rightarrow \geq n$  sets  
needed

Thm. (Ky Fan, 1952)

$S^{n-2}$  covered by open antipodal-free sets  $\Rightarrow \exists x, x'$   
antipodal  $x$  covered by  $\geq \lceil n/2 \rceil$ ,  $x'$  by  $\geq \lfloor n/2 \rfloor$  sets.

Zig-zag Thm. (Simonyi, T., 2006)

If  $\chi(G) \geq k$  for “topological reasons”, then  $\forall$  proper coloring of  $G$  contains a rainbow-colored  $K_{\lceil k/2 \rceil, \lfloor k/2 \rfloor}$

Corollary:  $\psi(G) \geq \lceil k/2 \rceil + 1$

Corollary: For Kneser, Schrijver, gen. Mycielski graphs:

$$\psi(G) \geq \lceil \chi(G)/2 \rceil + 1$$

Surprise: For large enough Schrijver and gen. Mycielski graphs:

$$\psi(G) \leq \lfloor \chi(G)/2 \rfloor + 2$$

Leaves a gap of 1 if  $\chi$  is even.

Thm. (Simonyi, T., Vrećica, 2009): For Kneser, Schrijver and gen. Mycielski graphs:

$$\psi(G) \geq \lfloor \chi(G)/2 \rfloor + 2$$

# Open

$\psi = \chi$  for all Kneser graphs?

Or  $\psi \approx \chi/2$  for some Kneser graphs?

Thm. (KPS 2005):  $\chi^*(G) \leq \psi(G) \leq \chi(G)$

$\chi^*(G) = \chi(G) \quad \Rightarrow \quad \psi(G)$  is determined

examples for  $\chi^*(G) \ll \chi(G)$

- Kneser graph
- Schrijver graphs
- (generalized) Mycielski graphs

$\chi(G)$  is found through  
“topological methods”

- shift graphs — topological methods fail
- ????



# shift graphs $H_m$

$$V(H_m) = \{(i,j) \mid 1 \leq i < j \leq m\}$$

$$E(H_m) = \{(i,j), (j,k) \mid 1 \leq i < j < k \leq m\}$$

Thm. (Simonyi, T., 2011)

$$\chi(H_m) - \psi(H_m) \leq 1$$
$$\chi(H_m) = \psi(H_m) \quad \text{if} \quad 2^k + 2^{k-1} < m \leq 2^{k+1}$$

Proof: Bollobás type inequalities

Is  $\chi(H_m) = \psi(H_m)$  for all  $m$ ?

(I'm sure.)

# Definitions

Erdős, Füredi, Hajnal, Komjáth, Rödl, Seress 1986

$$\psi(G) := \min_c \max_v |\{c(u) : \{u,v\} \in E(G)\}| + 1$$

The minimization is over all proper colorings  $c$ .

$\psi(G)$  is the minimum number of colors that must appear in the most colorful closed neighborhood of a vertex in any proper coloring.

Körner, Pilotto, Simonyi, 2005

$\vec{G}$  is a directed graph

$$\psi_d(\vec{G}) := \min_c \max_v |\{c(u) : (v,u) \in E(\vec{G})\}| + 1$$

The minimization is over all proper colorings  $c$  of the underlying graph  $G$ .

$\psi_d(\vec{G})$  is the minimum number of colors that must appear in the most colorful closed **outneighborhood** of a vertex in any proper coloring of  $G$ .

Obviously:  $\psi_d(\vec{G}) \leq \psi(G)$ .

Thm. (Simonyi, T., 2011)

If  $\chi(G) \geq k$  for “topological reasons”,  
then  $\psi_d(\vec{G}) \geq \lceil k/4 \rceil + 1$ .

from the Zig-zag theorem

Tight for large enough Schrijver and  
gen. Mycielski graphs.

# local chromatic number of $\vec{G}$ versus $G$

$$\psi_d(\vec{G}) \leq \psi(G)$$

= if both direction appears in  $\vec{G}$  of every edge of  $G$

Can = achieved always with an **orientation** (= no edge in both directions)?

Thm. (Simonyi, T., Zsbán, 2013)

$$\exists G \text{ with } \psi(G) = 4, \forall \text{ orientation } \psi_d(\vec{G}) = 3$$

How far can  $\psi_d(\vec{G})$  and  $\psi(G)$  be? **Very far.**

$\psi(H_m) \approx \log m$ , but for the “natural” orientation of  $\psi_d(H_m) = 2$

Thm. (KPS 2005):  $\chi^*(G) = \psi^*(G) \leq \psi(G)$

$\chi^*(H_m) < 4$ . How far can  $\psi_d(G)$  and  $\chi^*(G)$  be? **Not far.**

Thm. (STZs, 2013)

$$\sup \chi^*(G)/\psi_d(\vec{G}) = \sup \chi^*(G)/\psi_d^*(\vec{G}) = e$$

independently **Shanmugam-Dimakis-Langberg** with a worse constant

**OPEN:**

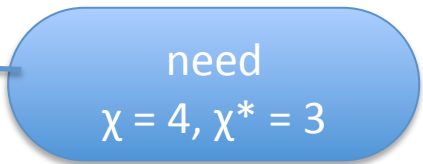
can the gap  
be arbitrarily  
large?

# graphs on surfaces

Mohar, Simonyi, T., 2013: Generalization of Youngs, Archdeakon-Hutchinson-Nakamoo-Negami-Ota, Mohar-Seymour results from chromatic to local chromatic number.

+ graph on genus 5 surface with  $\psi \neq \chi$ .

**OPEN:** planar graph with  $\psi \neq \chi$  ?



need  
 $\chi = 4, \chi^* = 3$