# Local chromatic number
## of graphs and digraphs

Gábor Tardos
Rényi Institute, Budapest

a survey

Budapest, Erdős 100, July 3, 2013

# Definition

## Erdős, Füredi, Hajnal, Komjáth, Rödl, Seress 1986

$\psi(G) := \min_c \max_v |\{c(u) : \{u,v\} \in E(G)\}| + 1$

The minimization is over all proper colorings *c.*

$\psi(G)$ is the minimum number of colors that must appear in the most colorful closed neighborhood of a vertex in any proper coloring.

Obviously: $\psi(\mathrm{G}) \leq \chi(\mathrm{G})$.

always = ?

Thm. (EFHKRS 1986): $\forall k, \ \exists G : \psi(G) = 3, \chi(G) > k$.

# revisiting local chromatic number
## Körner, Pilotto, Simonyi, 2005

+ directed version

related to "Sperner capacity" in information theory (KPS) + an information transmission problem (Shanmugam-Dimakis-Langberg 2013)

+ study of $\psi$ and $\psi_d$ as a graph parameters, e.g.:

Thm. (KPS 2005): $\chi^*(G) = \psi^*(G) \leq \psi(G)$

$\chi^*$ = fractional chromatic number

# Thm. (KPS 2005): $\chi^*(G) \le \psi(G) \le \chi(G)$

$\chi^*(G) = \chi(G)$ $\Rightarrow$ $\psi(G)$ is determined

examples for $\chi^*(G) << \chi(G)$

- Kneser graph
- Schrijver graphs
- (generalized) Mycielski graphs

} $\chi(G)$ is found through "topological methods"

- shift graphs
- ????

# topological method

started by Lovász, 1978

graph $G$ → complex B(G)
(topological space)

$\chi(G) \geq k$ ⇐ connectivity / dimension / etc of $B(G)$

↑

often using the Borsuk-Ulam theorem

Thm. (Lusternik-Schnirelmann, 1934 ≈ B-U)

$S^{n-2}$ covered by open antipodal-free sets ⇒ ≥ $n$ sets needed

Thm. (Ky Fan, 1952)

$S^{n-2}$ covered by open antipodal-free sets ⇒ $\exists\, x,x'$ antipodal $x$ covered by $\geq \lceil n/2 \rceil$, $x'$ by $\geq \lfloor n/2 \rfloor$ sets.

Zig-zag Thm. (Simonyi, T., 2006)
If $\chi(G)\geq k$ for "topological reasons", then $\forall$ proper coloring of $G$ contains a rainbow-colored $K_{\lceil k/2\rceil,\lfloor k/2\rfloor}$

Corollary: $\psi(G) \geq \lceil k/2 \rceil + 1$
Corollary: For Kneser, Scrijver, gen. Mycielski graphs:

$$\psi(G) \geq \lceil \chi(G)/2 \rceil + 1$$

Surprise: For large enough Schrijver and gen. Mycielski graphs:

$$\psi(G) \leq \lfloor \chi(G)/2 \rfloor + 2$$

Leaves a gap of 1 if $\chi$ is even.

Thm. (Simonyi, T., Vrećica, 2009): For Kneser, Schrijver and gen. Mycielski graphs:

$$\psi(G) \geq \lfloor \chi(G)/2 \rfloor + 2$$

# Open

$\psi = \chi$ for all Kneser graphs?

Or $\psi \approx \chi/2$ for some Kneser graphs?

# Thm. (KPS 2005): $\chi^*(G) \leq \psi(G) \leq \chi(G)$

$\chi^*(G) = \chi(G)$ $\Rightarrow$ $\psi(G)$ is determined

examples for $\chi^*(G) << \chi(G)$

- Kneser graph
- Schrijver graphs
- (generalized) Mycielski graphs

} $\chi(G)$ is found through "topological methods"

- shift graphs — topological methods fail
- ????

# shift graphs $H_m$

$V(H_m) = \{(i,j) \mid 1 \le i < j \le m\}$
$E(H_m) = \{(i,j),(j,k) \mid 1 \le i < j < k \le m\}$

Thm. (Simonyi, T., 2011)

$$\chi(H_m) - \psi(H_m) \le 1$$

$$\chi(H_m) = \psi(H_m) \quad \text{if} \quad 2^k + 2^{k-1} < m \le 2^{k+1}$$

Proof: Bollobás type inequalities

Is $\chi(H_m) = \psi(H_m)$ for all $m$?
(I'm sure.)

# Definitions

Erdős, Füredi, Hajnal, Komjáth, Rödl, Seress 1986

$\psi(G) := \min_c \max_v |\{c(u) : \{u,v\} \in E(G)\}| + 1$
The minimization is over all proper colorings *c*.

$\psi(G)$ is the minimum number of colors that must appear in the most colorful closed neighborhood of a vertex in any proper coloring.

Körner, Pilotto, Simonyi, 2005

$\vec{G}$ is a directed graph

$\psi_d(\vec{G}) := \min_c \max_v |\{c(u) : (v,u) \in E(\vec{G})\}| + 1$
The minimization is over all proper colorings *c* of the underlying graph *G*.

$\psi_d(\vec{G})$ is the minimum number of colors that must appear in the most colorful closed outneighborhood of a vertex in any proper coloring of *G*.

Obviously: $\psi_d(\vec{G}) \leq \psi(G)$.

**Thm.** (Simonyi, T., 2011)

If $\chi(G) \geq k$ for "topological reasons", then $\psi_d(\vec{G}) \geq \lceil k/4 \rceil + 1$.

from the Zig-zag theorem

Tight for large enough Schrijver and gen. Mycielski graphs.

# local chromatic number of $\vec{G}$ versus $G$

$\psi_d(\vec{G}) \leq \psi(G)$

= if both direction appears in $\vec{G}$ of every edge of $G$

Can = achieved always with an orientation (= no edge in both directions)?

Thm. (Simonyi, T., Zsbán, 2013)

$\exists\, G$ with $\psi(G) = 4$, $\forall$ orientation $\psi_d(\vec{G}) = 3$

OPEN:
can the gap
be arbitrarily
large?

How far can $\psi_d(\vec{G})$ and $\psi(G)$ be? Very far.

$\psi(H_m) \approx \log m$, but for the "natural" orientation of $\psi_d(\vec{H}_m) = 2$

Thm. (KPS 2005): $\chi^*(G) = \psi^*(G) \leq \psi(G)$

$\chi^*(H_m) < 4$. How far can $\psi_d(G)$ and $\chi^*(G)$ be? Not far.

Thm. (STZs, 2013)

$\sup \chi^*(G)/\psi_d(\vec{G}) = \sup \chi^*(G)/\psi_d^*(\vec{G}) = e$

independently Shanmugam-Dimakis-Langberg with a worse constant

# graphs on surfaces

Mohar, Simonyi, T., 2013: Generalization of Youngs, Archdeakon-Hutchinson-Nakamoo-Negami-Ota, Mohar-Seymour results from chromatic to local chromatic number.

+ graph on genus 5 surface with $\psi \neq \chi$.

OPEN: planar graph with $\psi \neq \chi$ ?

need
$\chi = 4$, $\chi^* = 3$