Erdős-type theorems for billiard models.

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ERDŐS $2^2.5^2$
Budapest, July 3, 2013

Erdős'start with probability theory and RW's

- 1939: P. Erdős, A. Wintner: Additive arithmetical functions and statistical independence (Prob. Number Theory)
- 1939: P. Erdős, M. Kac: On the Gaussian law of errors in the theory of additive functions (Prob. Number Theory)
- 1942: P. Erdős: On the law of the iterated logarithm (Random Walks)
- 1946: P. Erdős, M. Kac: On certain limit theorems of the theory of probability (Invariance principle)

Statistical physics

A main goal of statistical physics: macroscopic equations from microscopic assumptions (i. e. in classical physics: from Newtonian mechanics)

- Diffusion: convergence to Wiener process (or to Ornstein-Uhlenbeck or else) of a particle in gas or fluid
- Understanding heat conduction
- Effect of local impurities in a crystal

1905, **Einstein**: (Physicist's) Derivation of heat equation
1921, **Wiener**: mathematical model of Brownian motion

Role of RW models

- Derivation of macro behavior from stochastic models is 'easier' (from deterministic ones often not done yet!)
- Ideas used at stochastic models can be used or are instructive at deterministic ones

'Simplest' deterministic models: billiards and Lorentz process

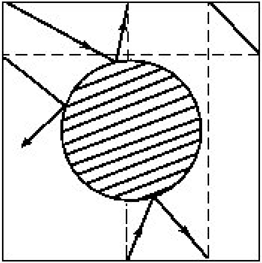
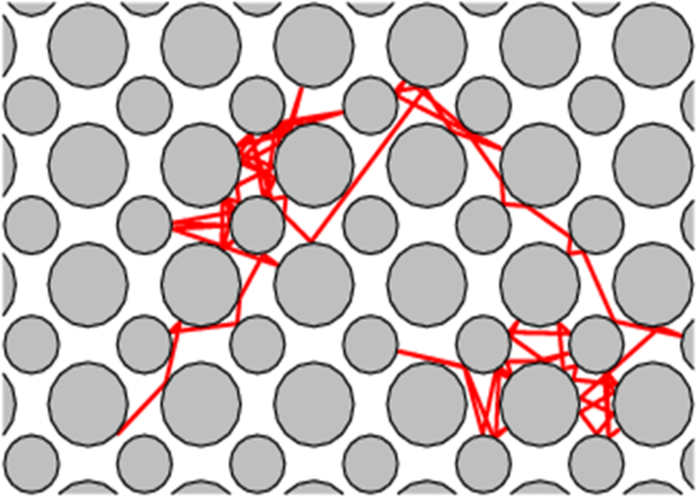


Figure: Sinai-billiard on two-torus

Periodic Lorentz Process



Lorentz process - billiard dynamics (uniform motion + specular reflection) (Ω, T, μ)

- $\hat{Q} = \mathbb{R}^d \setminus \cup_{i=1}^{\infty} O_i$ is the configuration space of the Lorentz flow (the billiard table), where the closed sets O_i are pairwise disjoint, strictly convex with $\mathcal{C}^3-$smooth boundaries
- $\Omega = Q \times S_+$ is the phase space for the discrete time map T (where $Q = \partial\hat{Q}$ and S_+ is the hemisphere of outgoing unit velocities)
- μ the T-invariant (infinite) Liouville-measure on Ω (actually Lebesgue $\otimes$ Lebesgue)

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Periodic Lorentz $\longrightarrow$ Sinai Billiard

If the scatterer configuration $\{O_i\}_i$ is $\mathbb{Z}^d$-**periodic**, then the corresponding dynamical system is $(\Omega_{per} = Q_{per} \times S_+, T_{per}, \mu_{per})$.
Then it makes sense to **factorize** it by $\mathbb{Z}^d$ to obtain a **Sinai billiard** $(\Omega_0 = Q_0 \times S_+, T_0, \mu_0)$. The natural projection $\Omega \to Q$ (for Ω_{per} or for Ω_0) is denoted by π_q.

Finite horizon (FH) versus *infinite horizon* (∞H)
Sinai-billiard is a hyperbolic dynamical system (like geodesic on negative curvature). BUT it is singular!

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Diffusion

Definition

Assume $\{q_n \in \mathbb{R}^d | n \geq 0\}$ is a random trajectory. Then its *diffusively scaled variant* $\in C[0,1]$ is: for $N \in \mathbb{Z}_+$

- $W_N(\frac{j}{N}) = \frac{q_j}{\sqrt{N}} \qquad (j = 1, 2, \dots, N)$
- and otherwise $W_N(t) \quad (t \in [0,1])$ is its piecewise linear, continuous extension.

Bunimovich-Sinai, 1981: $W_N(t) \Longrightarrow W(t)$ as $N \to \infty$.
(cf. Erdős-Kac, 1946 $\Longrightarrow$ Donsker,1951 & Prokhorov, 1954)

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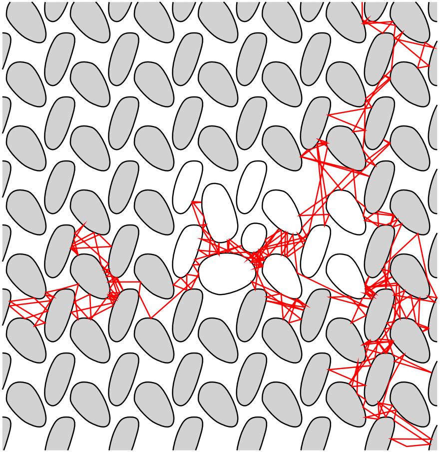
Sinai's question, 1981: Locally perturbed periodic Lorentz process?



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Method: $d = 2$

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Dolgopyat-Sz.-Varjú. I. 2008: Recurrence properties of periodic Lorentz process

Let S_n be the location of the (periodic) Lorentz particle after n collisions.
Let $m(S) = m \in \mathbb{Z}^2$ if $S \in Q_m$.
The first hitting of 0-th cell

$$\tau = \min\{n > 0 : m(S_n) = 0\} \quad (i.\ e.\ \tau : \Omega \to \mathbb{N})$$

Theorem

There is a constant $\mathbf{c}$ *such that* $\mu_0(\tau > n) \sim \frac{\mathbf{c}}{\log n}$.

Dolgopyat-Sz.-Varjú. I. 2008, continued

Let $N_n(x) = \mathrm{Card}(k \leq n : m(S_k) = 0)$.

Theorem

Assume x is distributed according to μ_0.
Then $\frac{cN_n}{\log n}$ converges weakly to a mean 1 exponential distribution.

The previous two theorems are analogues of Erdős-Taylor, 1960.
The next one is analogue of that of Darling-Kac, 1957.

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Let t_m denote the random variable $\tau(x)$ under the condition that x starts from the cell m (i. e. distributed according to μ_m).

Theorem

As $|m| \to \infty$, $\log t_m / 2 \log |m|$ *converges weakly to*

$$\xi = 1/U$$

where U *is a uniform random variable on* $[0,1]$.

Particular case of Mittag-Leffler distribution.
In particular: $t_m \asymp |m|^2$

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Range of RW

$$E_d(n) = \mathbb{E}(\#\text{ sites visited by RW during n steps})$$

Dvoretzky-Erdős, Some problems on random walk in space, 2nd Berkeley Symp. (1950)

Theorem

$$E_d(n) \asymp \begin{cases} \frac{\pi n}{\log n}(1+o(1)), & d=2 \\ n\gamma_d(1+o(1)), & d>2 \end{cases}$$

Question 1(Sz. 2006): $\mathbb{E}(\#\text{ cells visited by LP during n steps})$

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F. Pène: Asymptotic of the number of obstacles visited by the planar Lorentz process, DCDS(A), 2009

P. Nándori: Number of distinct sites visited by a RW with internal states, PTh&RF, 2011.

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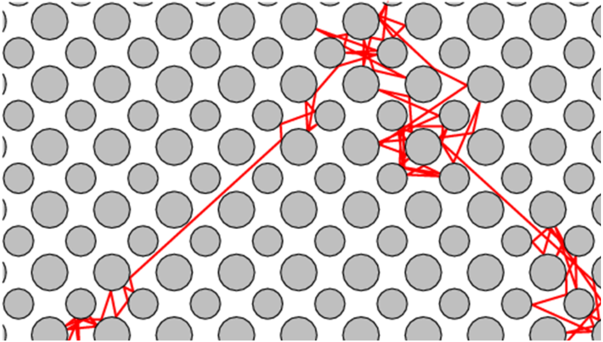
Self-intersections

Erdős-Taylor: Some intersection properties of random walk paths, Acta Math. Acad. Sci. Hung. (1960)

F. Pène: Self-intersections of trajectories of Lorentz process, 2013

∞H



Sz.-Varjú, 2007: As $N \to \infty$ $\frac{W_N}{\sqrt{N \log N}} \Longrightarrow W$

Analogous (periodic) RW, d = 1: $Prob(X_i = n) \sim \frac{const.}{n^3}$

Here $\mathbb{E}(X_i^2) = \infty$ but $\mathbb{E}(X_i^\alpha) < \infty$ $if \alpha < 2$

∞H

Question: locally perturbed ∞H periodic Lorentz process?

Paulin-Sz. 2010: $\frac{W_N}{\sqrt{N \log N}} \Longrightarrow W$ also holds for the locally perturbed RW of above (i. e. $\frac{c}{n^3}$) type.

Nándori, 2011: Recurrence properties of heavy tailed RW.

(Moreover, # of through-crossings of the origin for the RW above is $O(n^{1/6})$.)

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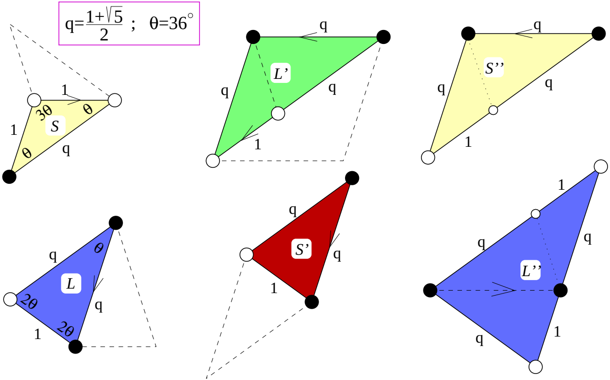
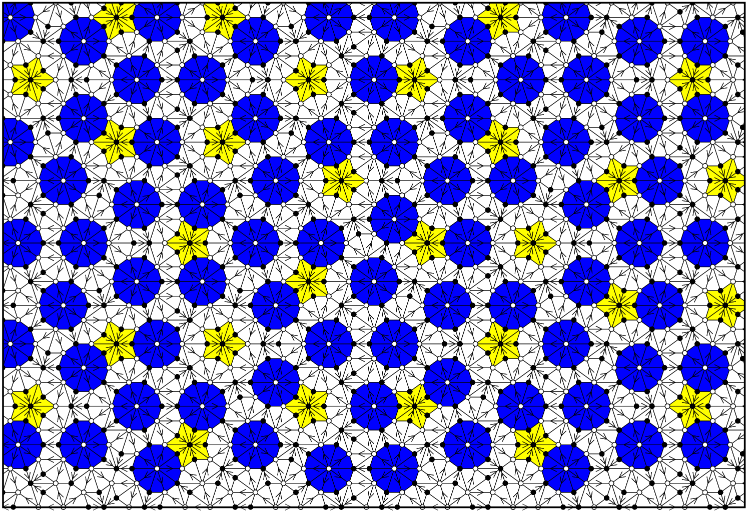
Prototiles of Penrose tiling



Penrose tiling of the plane



Global deviation from periodicity: Penrose-Lorentz process

Construct a planar finite horizon Lorentz-process based on the Penrose tiling: at each vertex of the Penrose tiling one puts identical circular scatterers. (The tiles with their scatterers inherit the symmetry of the Penrose tiling.) Call it a Penrose-Lorentz process.

Conjecture, Sz., 2006: By selecting the initial phase point of the Penrose-Lorentz process according to a probability measure absolutely continuous wrt to the Liouville measure, the diffusively scaled variant $W_N(t)$ of the Penrose-Lorentz trajectory converges weakly to a non-degenerate, rotation-invariant Wiener process.

Results for Penrose RW

M. Kunz, 2000: under the condition that harmonic coordinates exist, $\frac{S_n}{\sqrt{n}}$ is asymptotically normal with zero mean and a rotation invariant covariance matrix.

General results for RWs on graphs (Delmotte, 1999 and Hambly-Kumagai, 2004) combined with recent observation of Solomon, 2008 provide the asymptotic normality unconditionally (oral communication by A. Telcs).