Paul Erdős's results and influence in the theory of integer partitions

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In what follows I am dealing with some statistical properties of integer partitions which are proved or inspired by Paul Erdős.

Let Π be a generic "unrestricted" partition of the positive integer n, i.e., a representation of n as the sum of any number of positive integral parts arranged in descending order of magnitude:

$$\Pi: \quad \lambda_1 + \lambda_2 + ... + \lambda_m = n, \quad \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m (\geq 1),$$
$$\lambda_j\text{'s integers}, \quad m = m(\Pi).$$

Their number $p(n)$ has the following generating function. For $z \in C, |z| < 1$,

$$1 + \sum_{n=1}^{\infty} p(n) z^n = \prod_{\nu=1}^{\infty} \frac{1}{1 - z^\nu}.$$

Moreover,

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right)$$

according to (the simplest form of) a theorem of Hardy and Ramanujan from 1918.

[G. H. Hardy and S. Ramanujan, Asymptotic formulae in combinatory analysis, *Proc. London Math. Soc. (2)* **XVII** (1918), 75–115.]

By the words of Turán [P. Turán, The fiftieth anniversary of Pál Erdős, *Mat. Lapok* **14** (1963), 1–28 (in Hungarian); *Collected Papers of Paul Turán*, Akadémiai Kiadó, Budapest, 1990, Vol. **2**, 1493–1516],

"Erdős, consequently carrying through his program, here also gauged the power of "elementary" methods: in a paper published in the Annals of Math. in 1942 he showed that this formula — except the factor $\frac{1}{4\sqrt{3}}$ — lies within the range of "elementary" methods. ... Erdős added two further interesting contributions to the partition problem. With J. Lehner in 1941 in the Duke Journal he proved that — like Hardy and Ramanujan found for the distribution of the

prime factors of integers — "almost all" additive representations of a positive integer n contain "approximately" $\frac{1}{\pi}\sqrt{\frac{3}{2}}\sqrt{n}\log n \stackrel{\text{def}}{=} A(n)$ summands."

[P. Erdős, On an elementary proof of some asymptotic formulas in the theory of partitions, *Ann. of Math. (2)* **43** (1942), 437–450; P. Erdős and J. Lehner, The distribution of the number of summands in the partitions of a positive integer, *Duke Math. J.* **8** (1941), 335–345]

Thinking of the associate (or conjugate) partitions, the same holds for the maximal summand: If $\omega(n)\nearrow\infty$ arbitrarily slowly then

$$m = \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n + O(\sqrt{n}\,\omega(n))$$

and

$$\lambda_1 = \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n + O(\sqrt{n}\,\omega(n))$$

for almost all unrestricted Π's, i.e., with the exception of $o(p(n))$ partitions of n at most. Erdős and Lehner also proved that, with any real constant c,

$$\lambda_1 \le \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n + \frac{\sqrt{6}}{\pi}\sqrt{n}\cdot c$$

for

$$\left(\exp\left(-\frac{\sqrt{6}}{\pi}e^{-c}\right) + o(1)\right)p(n)$$

unrestricted Π's of n. [This is a doubly exponential or extreme-value distribution.] According to the result of Szekeres from 1987 the λ_1-distribution and the m-distribution are asymptotically independent in the range $[0.51A(n), 2A(n)]$ but there are correlations in the range $[0.5A(n), A(n)^{1.49}]$.

[G. Szekeres, Asymptotic distribution of partitions by number and size of parts, in: *Coll. Math. Soc. J. Bolyai*, **51** (*Number Theory*, Budapest, 1987), 527–538]

As to the other contribution mentioned by Turán, cite again.

"Let $p_k(n)$ be the number of partitions containing exactly k summands and for a given n define $k_0(n)$ by

$$p_{k_0}(n) = \max_k p_k(n).$$

In 1946 Erdős showed that for $n \to \infty$ we have

$$k_0(n) = A(n) + \frac{2}{\pi}\sqrt{\frac{3}{2}}\log\frac{\sqrt{6}}{\pi}\sqrt{n} + o(\sqrt{n}).$$

Later Szekeres proved that this $k_0(n)$ is unique, namely for a fixed (large) *n, $p_k(n)$ is increasing for $k \leq k_0$ and decreasing later."*

[P. Erdős, On some asymptotic formulas in the theory of partitions, *Bull. Amer. Math. Soc.* **52** (1946), 185–188; G. Szekeres, Some asymptotic formulae in the theory of partitions, II, *Quart. J. Math., Oxford Ser. (2)* **4** (1953), 96–111]

Thus, Erdős and Szekeres proved two conjectures of Auluck, Chowla, and Gupta.

[F. C. Auluck, S. Chowla, and H. Gupta, On the maximum value of the number of partitions of n into k parts, *J. Indian Math. Soc. (N. S.)* **6** (1942), 105–112]

Let $p_A(n)$ be the number of partitions of n into parts taken from the set $A \subseteq N^*$, repetitions being allowed. With Bateman in 1956 in Mathematika Erdős obtained conditions for A which imply that $p_A(n)$ is non-decreasing for large n and in Publ. Math. Debrecen they proved the monotonicity for $n \geq 1$ when A is the set of primes.

[P. T. Bateman and P. Erdős, Monotonicity of partition functions, *Mathematika* **3** (1956), 1–14; P. T. Bateman and P. Erdős, Partitions into primes, *Publ. Math. Debrecen* **4** (1956), 198–200]

In 1962 in Acta Arithmetica Erdős investigated the representation of large integers as sums of *distinct* summands *taken from a fixed set*. The result is weaker than Cassels's one, but the proof is elementary.

[P. Erdős, On the representation of large integers as sums of distinct summands taken from a fixed set, *Acta Arith.* **7** (1961/1962), 345–354; J.W.S. Cassels, On the representation of integers as the sums of distinct summands taken from a fixed set, *Acta Sci. Math. (Szeged)* **21** (1960), 111–124]

We remind to *distinct* summands *taken from a fixed set* above. At first, consider distinct summand from N^*.

Let Π^* be a generic "unequal" partition of the positive integer n, i.e., a representation of n as the sum of any number of distinct positive integral parts arranged in descending order of magnitude:

$$\Pi^*: \quad \mu_1 + \mu_2 + \ldots + \mu_{m^*} = n, \quad \mu_1 > \mu_2 > \ldots > \mu_{m^*} (\geq 1),$$

$$\mu_j\text{'s integers}, \quad m^* = m^*(\Pi^*).$$

Their number $q(n)$ has the following generating function. For $z \in C, |z| < 1$,

$$1 + \sum_{n=1}^{\infty} q(n) z^n = \prod_{\nu=1}^{\infty} (1 + z^\nu).$$

Moreover,

$$q(n) \sim \frac{1}{4n^{3/4}3^{1/4}} \exp\left(\frac{\pi}{\sqrt{3}}\sqrt{n}\right)$$

[G. H. Hardy and S. Ramanujan, Asymptotic formulae in combinatory analysis, *Proc. London Math. Soc. (2)* **XVII** (1918), 75–115.]

As to the *unequal* partitions, Erdős and Lehner also proved that, for almost all unequal Π^*'s, i.e., with the exception of $o(q(n))$ unequal partitions of n at most,

$$m^* = (1 + o(1)) \frac{2\sqrt{3}\log 2}{\pi}\sqrt{n}.$$

[P. Erdős and J. Lehner, The distribution of the number of summands in the partitions of a positive integer, *Duke Math. J.* **8** (1941), 335–345] After this Theorem 3.2 they stated without proof the followings: By sharper arguments we can obtain

Theorem 3.3. The number of unequal partitions of n in which the number of summands in a given partitions is less than

$$\frac{2\sqrt{3}\log 2}{\pi}\sqrt{n} + yn^{1/4}$$

is given by a Gaussian integral. (As to the maximal summand, it is known that

$$\mu_1 = (1 + o(1)) \frac{\sqrt{3}}{\pi}\sqrt{n}\log n$$

for almost all unequal Π^*'s.)

In the 1960s and the early seventies Erdős and Turán developed a statistical theory of the symmetric group S_n on n letters in a sequence of papers.

[P. Erdős and P. Turán, On some problems of a statistical group theory,
I, *Z. Wahrscheinlichkeitstheorie and verw. Gebiete* **4** (1965), 175–186;
II, *Acta Math. Acad. Sci. Hungar.* **18** (1967), 151–163;
III, *Acta Math. Acad. Sci. Hungar.* **18** (1967), 309–320;
IV, *Acta Math. Acad. Sci. Hungar.* **19** (1968), 413–435;
V, *Periodica Math. Hungar.* **1** (1971), 5–13;
VI, *J. Indian Math. Soc. (N. S.)* **34** (1970), 175–192;

VII, *Periodica Math. Hungar.* **2** (1972), 149–163]

In 1965, they proved in Part I that if $\omega(n) \nearrow \infty$ then the group theoretical order $O(P)$ of $P \in S_n$ satisfies the relation

$$O(P) = \exp\left\{\frac{1}{2}\log^2 n + O\left(\omega(n)\log^{3/2} n\right)\right\}$$

for *almost all* elements P of S_n (i.e., with the exception of $o(n!)$ P's at most as $n \to \infty$). The main point of the proof is that $O(P)$ is "essentially" the **product** of the *different* cycle-lengths in the canonical decomposition of P for *almost all* P's. The above factor $\omega(n)$ cannot be omitted. Moreover, Erdős and Turán proved in Part III that $O(P)$ shows a "logarithmic Gaussian distribution". More precisely, for any fixed real x,

$$\lim_{n\to\infty}\frac{1}{|S_n|}\left|\left\{P : P \in S_n, \log O(P) \le \frac{1}{2}\log^2 n + \frac{x}{\sqrt{3}}\log^{3/2} n\right\}\right| = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp(-t^2/2)dt.$$

Best in 1970 and Bovey in 1980 gave new proofs for this distribution theorem. Nicolas refined the distribution theorem by improving the product approximation.

[M. R. Best, The distribution of some variables on symmetric groups, *Proc. Kon. Ned. Akad. Wetensch.* A **73** (1970), 385–402 (*Indag. Math.* A **32** (1970), 385–402); J. D. Bovey, An approximate probability distribution for the orders of elements of the symmetric group, *Bull. London Math. Soc.* **12** (1980), 41–46; J.-L. Nicolas, Distribution statistique de l'ordre d'un élément du groupe symétrique, *Acta Math. Hungar.* **45** (1985), 69–84]

In 1981, Nicolas obtained a similar distribution theorem for the least common multiple of the degrees of the (different) irreducible polynomials in the standard factorization of the monic polynomials of degree n over a finite field. The corresponding product approximation was proved by Mignotte and Nicolas.

[J.-L. Nicolas, A Gaussian law on $F_q[X]$, in: *Coll. Math. Soc. J. Bolyai*, **34.** *Topics in Classical Number Theory* (Budapest, 1981), 1127–1162, North-Holland/Elsevier; M. Mignotte and J.-L. Nicolas, Statistique sur $F_q[X]$, *Ann. Inst. H. Poincaré Sect. B (N. S.)*, **19** (1983), 113–121]

In 1967, Dénes, Erdős, and Turán obtained a distribution theorem for the orders of the elements of the alternating group A_n on n letters. In 1973, Harris proved an analogous result for the symmetric semigroup S_n^* on n letters.

[J. Dénes, P. Erdős, and P. Turán, On some statistical properties of the alternating group of degree n, *L'Enseignement mathématique (2)* **15** (1969), 89–99; B. Harris, The asymptotic distribution of the order of elements in symmetric semigroups, *J. Combinatorial Theory* **15A** (1973), 66–74]

The mentioned sequence of papers by Erdős and Turán contains a number of *statistical* results on the arithmetical structure of $O(P)$ for $P \in S_n$, on the possible *different* values of $O(P)$ for $P \in S_n$, on the cardinalities of the conjugacy classes of S_n and on the common orders of the elements in a random conjugacy class of S_n. These results are closely connected with the *statistical* theory of *partitions* of various type and "weightings" (think of the number of conjugacy classes of S_n which is $p(n)$, the number of unrestricted partitions of n). E.g., the number of different values of $O(P)$ in $S_n \setminus \{1\}$ is equal to the total number of partitions of the integers $2, 3, ..., n$ into powers of different primes, i.e., the number of solution of

$$n \geq q_1^{\beta_1} + q_2^{\beta_2} + ..., \qquad 2 \leq q_1 < q_2 < ... \quad (q_j\text{'s primes}).$$

In Part IV Erdős and Turán obtained

$$\exp\left\{\frac{2\pi}{\sqrt{6}}\sqrt{\frac{n}{\log n}} + O\left(\frac{\sqrt{n}\log\log n}{\log n}\right)\right\}$$

for the above number of solutions and proved that the number of summands is

$$\frac{2\sqrt{6}}{\pi}\log 2\sqrt{\frac{n}{\log n}} + O\left(\sqrt{n}\log^{-0.73} n\right)$$

in *almost all* solutions. This implies that *almost all* of the possible *different* $O(P)$-values are of the form

$$\exp\left\{(1 + o(1))\,\frac{\sqrt{6}\log 2}{\pi}\sqrt{n\log n}\right\}$$

which is very large in comparison with the value

$$\exp\left\{\frac{1}{2}\log^2 n + O\left(\omega(n)\log^{3/2} n\right)\right\}$$

and is roughly square root of Landau's maximum $\exp\{(1+o(1))\sqrt{n\log n}\}$. [E. Landau, Über die Maximalordnung der Permutationen gegebenen Grades, *Arch. Math. Phys. (3)* **5** (1903), 92–103]

Consequently, most of the different $O(P)$-values are "almost" as large as possible, but these values belong to "few" P's.

For other details and related results we refer to [P. Erdős and M. Szalay, Note to Turán's papers on the statistical theory of groups and partitions, in: *Collected Papers of Paul Turán*, Akadémiai Kiadó, Budapest, 1990, Vol. **3**, 2583–2603; E. Schmutz, Proof of a conjecture of Erdős and Turán, *J. Number Theory* **31** (1989), 260–271; P. Erdős and M. Szalay, On some problems of the statistical theory of partitions, in: *Coll. Math. Soc. J. Bolyai*, **51** (*Number Theory*, Budapest, 1987), 93–110; W. M. Y. Goh and E. Schmutz, The expected order of a random permutation, *Bull. London Math. Soc.* **23** (1991), 34–42; H.-K. Hwang, Limit theorems for the number of summands in integer partitions, *J. Combin. Theory Ser. A* **96** (2001), 89–126; E. Manstavičius, The Berry–Esseen bound in the theory of random permutations, *Ramanujan J.* **2** (1998), 185–199; E. Manstavičius, On random permutations without cycles of some lengths, *Periodica Math. Hungar.* **42** (2001), 37–44; A. I. Pavlov, On the Erdős–Turán theorem on the logarithm of an order for a random permutation, *Dokl. Akad. Nauk* **350** (1996), 170–173 (in Russian); V. Zakharovas, Distribution of the logarithm of the order of a random permutation, *Lithuanian Math. J.* **44** (2004), 296–327]

The problem

$$n \geq q_1^{\beta_1} + q_2^{\beta_2} + ..., \qquad 2 \leq q_1 < q_2 < ... \quad (q_j\text{'s primes})$$

led to a general result when the summands are taken from a given sequence

$$A \;\; : \;\; 0 < a_1 < a_2 < ...$$

of integers. In 1969, Erdős and Turán proved the following general theorem by supposing *only* an asymptotic requirement on the counting function

$$\Phi_A(x) = \sum_{a_\nu \leq x} 1.$$

If α and β are real constants, $0 < \alpha \leq 1$ and

$$\lim_{x\to+\infty} \Phi_A(x) x^{-\alpha}\log^\beta x = B$$

then in *almost all* solutions of

$$n \geq a_{i_1} + a_{i_2} + ..., \qquad 1 \leq i_1 < i_2 < ...$$

the number of summands is

$$(1+o(1))C_1(\alpha,\beta,B)n^{\alpha/(\alpha+1)}\log^{-\beta/(\alpha+1)}n \qquad (n\to\infty).$$

Note that a somewhat stronger asymptotic requirement on $\Phi_A(x)$ and a not too strong lower bound on the number of solutions yield an analogous result for

$$n = a_{i_1} + a_{i_2} + ..., \qquad 1 \le i_1 < i_2 < ...$$

too. [P. Erdős and P. Turán, On some general problems in the theory of partitions, I, *Acta Arith.* **18** (1971), 53–62]

A result of de Bruijn was generalized by various authors concerning a periodic or almost periodic term in the asymptotic behaviour of $p_A(n)$ when $\liminf \frac{\log a_\nu}{\nu} > 0$. Erdős and Richmond proved by an example that this may happen for sequences that satisfy $a_\nu \sim \nu$ and considered an analogous phenomena for partitions into primes. They also considered corresponding results for $q_A(n)$.

[N. G. de Bruijn, On Mahler's partition problem, *Proc. Nederl. Akad. Wetensch.* **51** (1948), 659–669 (*Indag. Math.* **10** (1948), 210–220); P. Erdős and B. Richmond, Concerning periodicity in the asymptotic behaviour of partition functions, *J. Austral. Math. Soc. Ser. A* **21** (1976), 447–456]

In the mentioned 1942 Annals of Math. paper Erdős obtained *logarithmic asymptotic* results for the number of partitions of n into summands (resp. distinct summands) relatively prime to n. In 1978, Erdős and Richmond obtained *asymptotic* formulae. [P. Erdős and B. Richmond, On partitions of N into summands coprime to N, *Aequationes Math.* **18** (1978), 178–186]

We have to mention some more uncoventional partition problems of Erdős. For an irrational number $\alpha > 1$, let $a_\nu = [\nu\alpha]$ in A and $\gamma = \alpha - [\alpha]$. Erdős and Richmond obtained asymptotic formulae for $p_A(n)$ and $q_A(n)$ for *almost all* γ. In 1979, Erdős and Loxton estimated the number of partitions of n of the form $n = a_1 + a_2 + ... + a_k$ where $a_1|a_2|...|a_k$.

[P. Erdős and B. Richmond, Partitions into summands of the form $[m\alpha]$, *Proceedings of the Seventh Manitoba Conference on Numerical Mathematics and Computing (Univ. Manitoba, Winnipeg, Man., 1977), Congress. Numer., XX*, 371–377 (Utilitas Math., Winnipeg, Man., 1978); P. Erdős and J. H. Loxton, Some problems in partitio numerorum, *J. Austral Math. Soc. Ser. A* **27** (1979), 319–331]

We can also obtain an asymptotic formula for the number of partitions of n into unequal parts $\ge m$ when $m \le n^{3/8-\varepsilon}$.

[P. Erdős, J.-L. Nicolas, and M. Szalay, Partitions into parts which are unequal and large, in: *Number Theory (Ulm, 1987)*, 19–30, *Lecture Notes in Math.* **1380**, Springer, New York, 1989]

By means of partitions, it is also shown that almost all conjugacy classes of the alternating group A_n contain a pair of elements which generate A_n.

[L. B. Beasley, J. L. Brenner, P. Erdős, M. Szalay, and A. G. Williamson, Generation of alternating groups by pairs of conjugates, *Periodica Math. Hungar.* **18** (1987), 259–269]

J. Dénes raised the following interesting problem. What is the number of pairs (Π_1, Π_2) of partitions of n which do not have equal **subsums** (apart from the complete subsum n)? Also the investigation of common **summands** led Turán to some unexpected phenomena. He proved that *almost all* pairs of partitions of n contain

$$\left(\frac{\sqrt{6}}{4\pi} - o(1)\right)\sqrt{n}\log n$$

common *summands* at least *with multiplicity*. One can imagine that this phenomenon is perhaps caused by certain summands of great multiplicity. This is not the "real" reason. Turán proved generalizations for k-tuples of (partitions resp.) **unequal** partitions of n.

[P. Turán, On some connections between combinatorics and group theory, in: *Coll. Math. Soc . J. Bolyai*, **4** (*Combinatorial Theory and Its Applications*, Balatonfüred, 1969), 1055–1082; P. Turán, Combinatorics, partitions, group theory, in: *Colloquio Int. s. Teorie Combinatorie* (Roma, 3–15 settembre 1973), Roma, Accademia Nazionale dei Lincei, 1976. Tomo II, 181–200; P. Turán, On a property of partitions, *J. Number Theory* **6** (1974), 405–411]

Another approach to the original problem of the subsums would be — as Turán proposed to Erdős — the investigation of the integers which can be represented by subsums. With Erdős we proved that the number of partitions of n which represent all integers k of the interval $[1, n]$ as subsum is

$$\left(1 - \frac{\pi}{\sqrt{6n}} + O\left(\frac{\log^{30} n}{n}\right)\right)p(n),$$

consequently, *almost all partitions* of n represent *all* integers of $[1, n]$ as subsums. The analogue of this assertion does not hold for unequal partitions (e.g., it easy to see that $k = 1$ cannot be represented in a positive percentage of the unequal partitions of n) but we obtained the following weaker result of similar type. Let k_0 be an integer with $1 \leq k_0 \leq n/2$. Then the unequal partitions of n represent all integers k of the interval $[k_0, n-k_0]$ as subsums apart from

$$\left(20\left(2/\sqrt{3}\right)^{-k_0} + O\left(n^{-1/10}\right)\right) q(n)$$

unequal partitions of n at most.

[P. Erdős and M. Szalay, On some problems of J. Dénes and P. Turán, in: *Studies in Pure Mathematics, To the Memory of Paul Turán*, Akadémiai Kiadó, Budapest, 1983, 187–212]

We say that a partition Π is "practical" if it represents all the integers $1, 2, ..., n$ by subsums. For the number $M(n)$ of nonpractical partitions of n we infer the following asymptotic relation

$$M(n) = \left(\frac{\pi}{\sqrt{6n}} + O\left(\frac{\log^{30} n}{n}\right)\right) p(n).$$

In 1987, Dixmier and Nicolas obtained an asymptotic expansion for $M(n)/p(n)$ in terms of powers of $n^{-1/2}$. In 1995, Erdős and Nicolas proved similar results for some cases when the parts are taken from special sets.

[J. Dixmier and J.-L. Nicolas, Partitions without small parts, in: *Coll. Math. Soc. J. Bolyai*, **51** (*Number Theory*, Budapest, 1987), 9–33; P. Erdős and J.-L. Nicolas, On practical partitions, *Collect. Math.* **46** (1995), 57–76]

Dixmier, Erdős, Nicolas, and Sárközy investigated the asymptotic behaviour of the number of partitions of n without a given subsum. In 1992, Erdős, Nicolas, and Sárközy solved the problem of J. Dénes from 1967 by obtaining an asymptotic expansion for the number of pairs of partitions of n which do not have nontrivial equal subsums:

$$2p(n)\left(1 + \alpha_1 n^{-1/2} + \alpha_2 n^{-1} + ... + \alpha_k n^{-k/2} + O\left(n^{-(k+1)/2}\right)\right).$$

[J. Dixmier, Sur les sous-sommes d'une partition *Mém. Soc. Math. France (N. S.)* **35** (1988), 70 pp. (1989); P. Erdős, J.-L. Nicolas, and A. Sárközy, On the number of partitions of n without a given subsum, I, *Discrete Math.* **75** (1989), 155–166; P. Erdős, J.-L. Nicolas, and A. Sárközy, On the number of partitions of n without a given subsum, II, in: *Analytic Number Theory (Allerton Park, IL, 1989)*, 205–234, *Progr. Math.* **85**, Birkhäuser Boston, Boston, MA, 1990; P. Erdős, J.-L. Nicolas, and A. Sárközy, On the number of pairs of partitions of n without common subsums, *Colloquium Math.* **63** (1992), 61–83]

I have to mention shortly some results concerning the distribution of summands. For $\log^6 n \le k \le \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n - 5\sqrt{n}\log\log n$,

$$\lambda_k = \left(1 + O(\log^{-1} n)\right) \frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp\left(-\pi k/\sqrt{6n}\right)}$$

uniformly with the exception of $O(p(n)/n)$ partitions of n.

[M. Szalay and P. Turán, On some problems of the statistical theory of partitions with application to characters of the symmetric group, III, *Acta Math. Acad. Sci. Hungar.* **32** (1978), 129–155]

For unequal partitions one can obtain analogous estimations, roughly, instead of

$$\frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp\left(-\pi k/\sqrt{6n}\right)}$$

with

$$\frac{\sqrt{12}}{\pi}\sqrt{n}\log\frac{1}{\exp\left(\pi k/\sqrt{12n}\right)-1}.$$

However, the increasing order (with $\mu'_k = \mu_{m^*+1-k}$) is more interesting. With Erdős we proved that, for $\omega(n)\nearrow\infty$ and $\omega(n) \le k \le \sqrt{n}/\omega(n)$, that $\mu'_k \sim 2k$ for almost all unequal partitions of n.

[P. Erdős and M. Szalay, On the statistical theory of partitions, in: *Coll. Math. Soc J. Bolyai*, **34** (*Topics in Classical Number Theory*, Budapest, 1981), 397–450]

Next, consider the graphical partition problem. A partition Π of an even integer n is said to be graphical if there exists a graph of m vertices with degree sequence $\{\lambda_1, ..., \lambda_m\}$. In 1982 H. Wilf conjectured that almost all Π are not graphical. One can use the Erdős–Gallai conditions. Erdős and Richmond, Rousseau and Ali obtained partial results. In 1999, Pittel confirmed Wilf's conjecture.

[P. Erdős and T. Gallai, Graphs with points of prescribed degrees (in Hungarian), *Mat. Lapok* **11** (1961), 264–274; P. Erdős and L. B. Richmond, On graphical partitions, *Combinatorica* **13** (1993), 57–63; C. Rousseau and F. Ali, On a conjecture concerning graphical partitions, *Congr. Numer.* **104** (1994), 150–160; B. Pittel, Confirming two conjectures about the integer partitions, *J. Combin. Theory Ser. A* **88** (1999), 123–135]

Finally, consider the distribution of summands in residue classes. With Dartyge and Sárközy we proved, e.g., that if $\omega(n)\nearrow\infty, d \le n^{(1/2)-\varepsilon}$ $(\varepsilon > 0)$, and

$$\frac{\omega(n)}{\log n} d \le r \le d$$

then, for all but $p(n)/\omega(n)$ unrestricted partitions of n,

$$\sum_{\substack{k \\ \lambda_k \equiv r(d)}} 1 = (1+o(1))\frac{\sqrt{6n}}{\pi d}\log(\frac{\sqrt{n}}{d}).$$

The condition on r can be explained by the fact that in a "random" unrestricted partition of n the $O(1)$ summands occur with frequency of order of magnitude $\sqrt{n}$. But we can drop the condition $\frac{\omega(n)}{\log n}d \leq r$ if we don't count the "small" parts:

$$\sum_{\substack{k \\ \lambda_k \equiv r(d) \\ \lambda_k > r}} 1 = (1+o(1))\frac{\sqrt{6n}}{\pi d}\log(\frac{\sqrt{n}}{d})$$

for almost all unrestricted partitions of n.

[C. Dartyge, A. Sárközy, and M. Szalay, On the distribution of the summands of partitions in residue classes, *Acta Math. Hungar.* **109** (2005), 215–237]