

PAUL ERDŐS
AND
GRAPH RAMSEY THEORY

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ETH and UCLA

RAMSEY THEOREM

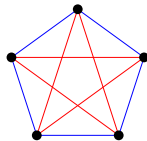
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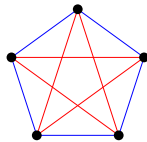


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THEOREM: (*Ramsey 1930*)

For all s, n , the Ramsey number $r(s, n)$ is finite.

DIAGONAL RAMSEY NUMBERS

THEOREM: (*Erdős 1947, Erdős-Szekeres 1935*)

$$2^{n/2} \leq r(n, n) \leq 2^{2n}.$$

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Lower bound: Color every edge randomly. Probability that a given set of n vertices forms a monochromatic clique is $2 \cdot 2^{-\binom{n}{2}}$.
Use the union bound.

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QUESTIONS: (*Erdős et al. 70's*)

- Can we strengthen Ramsey's theorem to show that the monochromatic clique has some additional structure?
- What controls the growth of $r(G)$ as a function of the number of vertices of G ?"
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Motivation: This will test the limits of current methods and may also lead to the development of new techniques that eventually would give better estimates for $r(n, n)$.

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THEOREM: (*Rödl 2003*)

There is a monochromatic clique with weight $\approx \log \log \log \log N$.

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Upper bound (Rödl):

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THEOREM: (*Conlon-Fox-S. 2013+*)

Maximum weight of monochromatic clique is $\Theta(\log \log \log N)$.

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THEOREM: (*Graham-Rödl-Ruciński 2000*)

$$2^{\Omega(d)} \leq c_d \leq 2^{O(d \log^2 d)}.$$

METHODS: LOCAL DENSITY

Key idea: Suppose that a red-blue edge-coloring of the complete graph K_N has the following property:

For any two disjoint sets X, Y of linear size there are at least $\varepsilon|X||Y|$ red edges between them (i.e., density of red $d(X, Y) \geq \varepsilon$).

If n/N is a sufficiently small constant, red contains copy of every n -vertex bounded degree G .

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Let G be an n -vertex graph with maximum degree d , then

$$r(G) \leq 2^{O(d \log d)} \cdot n.$$

If G is bipartite, then

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Remark: The second result is tight and gives the best known bound for Ramsey numbers of *binary cubes*. Cube Q_d has vertex set $\{0, 1\}^d$ and x, y are adjacent if x and y differ in exactly one coordinate.

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CONJECTURE: (*Burr-Erdős 1975*)

If G is a d -degenerate n -vertex graph, then $r(G)$ is linear in n .

RAMSEY NUMBERS OF DEGENERATE GRAPHS

Linear bound is known for:

- Bounded degree graphs (*Chvátal-Rödl-Szemerédi-Trotter 83*)
- Planar graphs and graphs drawn on bounded genus surfaces (*Chen-Schelp 93*)
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Remark: Best upper bound on $r(G)$ is $2^{c\sqrt{\log n}} \cdot n$ (*Fox-S 09*).

METHODS: DEPENDENT RANDOM CHOICE

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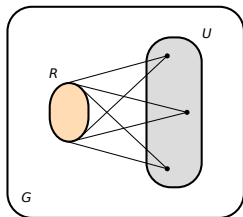
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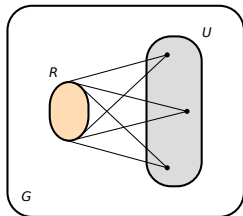


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If some set of d vertices has only *few* common neighbors, it is unlikely that all the members of R will be chosen among these neighbors. Hence we do not expect U to contain any such d vertices.

□

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Remark: This coloring also contains two sets X and Y such that, all edges in X and from X to Y have the same color and

$$|X| = t, \quad |Y| \geq k^{-20t/k} N.$$

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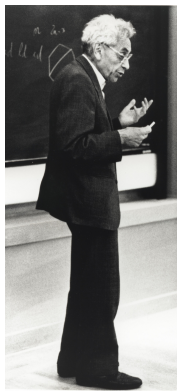
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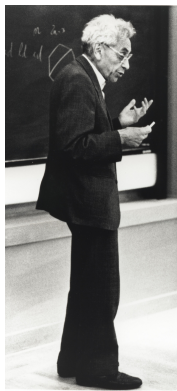
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Note $|U_2| \leq k_1\sqrt{m} \leq |X_2|$ and $\Delta(G_2) \leq \frac{2\sqrt{m}}{k_1}$.
- Apply above process recursively with $k_1 \ll k_2 \ll \dots$

SUMMARY



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These challenges, the fundamental nature of Graph Ramsey Theory and its tight connection with other areas will ensure that in the future this subject, started by Paul Erdős 40 years ago, will continue to play an essential role in the development of discrete mathematics.