Six Standard Deviations *Still* Suffice!

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Erdős Magic:

If a random object has a positive probability of being good than a good object MUST exist

Modern Erdős Magic:

If a randomized algorithm has a positive probability of producing a good object than a good object MUST exist

Working with Paul Erdős was like taking a walk in the hills. Every time when I thought that we had achieved our goal and deserved a rest, Paul pointed to the top of another hill and off we would go.
– Fan Chung

Eliminating Outliers

Six Standard Deviations Suffice

$S_1, \ldots, S_n \subseteq \{1, \ldots, n\}$

$\chi : \{1, \ldots, n\} \rightarrow \{-1 + 1\} = \{red, blue\}$

$\chi(S) := \sum_{j \in S} \chi(j)$, $\mathtt{disc(S)} = |\chi(\mathtt{S})| = |\mathtt{red} - \mathtt{blue}|$

Theorem (JS/1985): There *exists* χ

$$\mathtt{disc(S_i)} \leq 6\sqrt{\mathtt{n}} \text{ for all } 1 \leq \mathtt{i} \leq \mathtt{n}$$

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Theorem (Bansal/2010): Yes I can!

Theorem (Lovett, Meka/2012): We can too!

A Vector Formulation

$\vec{r_i} \in R^n$, $1 \leq i \leq n$, $|\vec{r_i}|_\infty \leq 1$

Initial $\vec{z} \in [-1, +1]^n$ (e.g.: $\vec{z} = \vec{0}$.)

Theorem: There exists $\vec{x} \in \{-1, +1\}^n$ with

$$|\vec{r_i} \cdot (\vec{x} - \vec{z})| \leq K\sqrt{n}$$

for *all* $1 \leq i \leq n$.

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$\vec{z} = \vec{0}$, $\vec{x}$ random. Problem: OUTLIERS!

Phase I

Find $\vec{x} \in [-1, +1]^n$ with all least $\frac{n}{2}$ at ± 1.

Idea: Start $\vec{x} \leftarrow \vec{z}$. Move $\vec{x}$ in a Controlled Brownian Motion.

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Set $L_j = [n^{-1/2}\vec{r_j}] \cdot [\vec{x} - \vec{z}]$

WANT: ALL $|L_j| \leq K$

Space V of allowable moves $\vec{y} = (y_1, \ldots, y_n)$

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KEY: $\vec{y}$ orthogonal to $\vec{r_j}$ for j with top $\frac{n}{4}$ $|L_j|$

The Random Move

$d = \dim(V) \geq \frac{n}{4} - 1 \sim \frac{n}{4}$.

Gaussian $\vec{g} = d^{-1/2}[n_1\vec{b_1} + \ldots + n_d\vec{b_d}]$, orthonormal $\vec{b_s}$

Move $\vec{x} \leftarrow \vec{x} + \delta\vec{g}$

Analysis

$|\vec{x}|^2 \leftarrow |\vec{x}|^2 + \delta^2$ so $T \le n\delta^{-2}$

L_j moves Gaussian, Variance $\le \delta^2 \frac{1}{d} \le \delta^2 \frac{4}{n}$

Total Variance ≤ 4. Martingale

$\Pr[|L_j| \ge K] \le 2e^{-K^2/8} \le 0.1$

SUCCESS: Fewer than $\frac{n}{5}$ j with $|L_j| \ge K$.

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SUCCESS implies that ALL $|L_j| \leq K$

Phase s

$m = 2^{1-s}n$. Start $\vec{z}$ with $\leq m$ coordinates frozen. End $\vec{x}$ with $\leq \frac{m}{2}$ coordinates frozen.

Effectively $|\vec{r_j}| \leq \sqrt{m}$

Would get $K\sqrt{m}$ but still have $n = m2^{s-1}$ vectors.

Actually get: $K\sqrt{m}\sqrt{s} = K\sqrt{n}\sqrt{s}2^{(1-s)/2}$

Converges!

Thank You!

It is six in the morning.
The house is asleep.
Nice music is playing.
I prove and conjecture.
– Paul Erdős, in letter to Vera Sós