# On the sum-product problem

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# The Sum-Product Problem

A conjecture of Erdős and Szemerédi states that if $A$ is a finite set of integers then the sum-set or the product-set should be large. The sum-set of $A$

$$A + A = \{a + b | a, b \in A\},$$

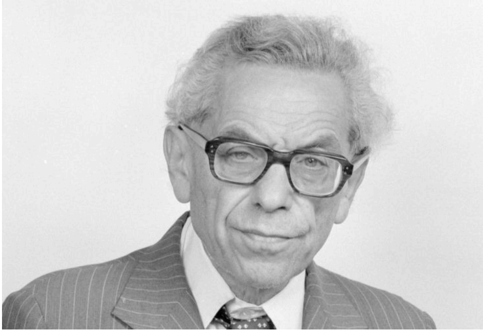
and the product set are defined in a similar way,

$$A \cdot A = \{ab | a, b \in A\}.$$

Erdős and Szemerédi conjectured that the sum-set or the product set is almost quadratic in the size of $A$, i.e.

$$\max(|A + A|, |A \cdot A|) \geq c|A|^{2-\delta}$$

for any positive $\delta$.



# The real case

$$\max\{|A+A|, |AA|\} \geq |A|^{1+\delta}$$

In a series of papers various lower bounds were find.

- $\delta \geq 1/31$ Nathanson (1996),
- $\delta \geq 1/15$ Ford (1997),
- $\delta \geq 1/4$ Elekes (1998),
- $\delta \geq 3/11$ S. (2003), and $\delta \geq 1/3$ S. (2009).

The last three bonds were proved for finite sets of real numbers. For complex numbers $\delta \geq 3/11$ was proved by Tardos and S. (2007) and $\delta \geq 1/3$ by Konyagin and Rudnev (2013).

# The finite field case, $\mathbb{F}_p$.

Bourgain, Katz, and Tao proved a nontrivial, $|A|^{1+\varepsilon}$, lower bound for the finite field case in 2004. Let $A \subset \mathbb{F}_p$ and $p^\alpha \leq |A| \leq p^{1-\alpha}$. Then there is an $\varepsilon > 0$ depending on $\alpha$ only, such that

$$\max(|A + A|, |A \cdot A|) \geq c|A|^{1+\varepsilon}.$$

(Konyagin showed that the lower bound, $p^\alpha \leq |A|$, is not needed.)
It is important that $p$ is prime, otherwise one could select $A$ being a subring in which case both the product set and the sum set are small, equal to $|A|$.

# Effective bounds

There are good sum-product estimates for the case, $|A| \gg p^{1/2}$. (Iosevich, Hart, and S. (2006), Garaev (2007), Granville and S.(2008)).

It follows from a construction by Ruzsa, that Garaev's bound is asymptotically the best possible for the range $|A| \geq p^{2/3}$.

Garaev's proof uses bounds on exponential sums. We are going to derive similar sum-product estimates using spectral bounds for graphs.

# The Sum-Product graph

The vertex set of the sum-product graph $G_{SP}$ is the Cartesian product of the multiplicative subgroup and the field,

$$V(G_{SP}) = \mathbb{F}_p^* \times \mathbb{F}_p.$$

Two vertices,

$$v_i = (a, b), v_j = (c, d) \in V(G_{SP}),$$

are connected by and edge,

$$(v_i, v_j) \in E(G_{SP}),$$

iff

$$ac = b + d.$$

# The Sum-Product graph

The adjacency matrix of $G_{SP}$ - denoted by $M$ - is symmetric, so all $p(p-1)$ eigenvalues are real, we can order them,

$$\mu_0 \geq \mu_1 \geq \ldots \geq \mu_{p^2-p-1}.$$

The second largest eigenvalue, $\lambda$, is defined as

$$\lambda = \max(\mu_1, |\mu_{p^2-p-1}|).$$

We are looking for a good upper bound on $\lambda$.

# The Sum-Product graph

For any two vertices, $v_i = (a, b)$ and $v_j = (c, d) \in V(G_{SP})$, if $a \neq c$ and $b \neq d$, then the vertices have exactly one common neighbor, $N(v_i, v_j) = (x, y) \in V(G_{SP})$.

The unique solution of the system

$$\begin{aligned} ax &= b + y \\ cx &= d + y \end{aligned} \quad \text{is given by} \quad \begin{aligned} x &= (b - d)(a - c)^{-1} \\ 2y &= x(a + c) - b - d. \end{aligned}$$

If $a = c$ or $b = d$, then the vertices, $v_i, v_j$, have no common neighbors.

# The Sum-Product graph

$$M^2 = J + (p-2)I - E,$$

where $J$ is the all-one matrix, $I$ is the identity matrix, and $E$ is the ”error matrix”, the adjacency matrix of the graph, $G_E$, where for any two vertices, $v_i = (a, b)$ and $v_j = (c, d) \in V(G_{SP})$, $(v_i, v_j) \in E(G_E)$ iff $a = c$ or $b = d$.

$$
M = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
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1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
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1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
$$

$$M^2 = \begin{bmatrix}
4 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 4 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 4 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 4 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 4 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
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1 & 0 & 1 & 1 & 1 & 0 & 4 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 4 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
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1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
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1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 4 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
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0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 4 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 4 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 4 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 4 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 4
\end{bmatrix}$$

$$E = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
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0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
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1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
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0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}$$

$$M^2 = J + (p-2)I - E,$$

Multiply both sides of the matrix equation above by $\overrightarrow{v_\lambda}$.

$$(\lambda^2 - p + 2)\overrightarrow{v_\lambda} = E\overrightarrow{v_\lambda}.$$

$E$ is a $2p-3$-regular graph, so any eigenvalue of $E$ has magnitude at most $2p-3$.

$$|\lambda^2 - p + 2| \leq 2p - 3,$$

$$\lambda < \sqrt{3p}.$$

# The discrepancy bound

$S, T \subset V(G_{SP})$,

$$\left| e(S,T) - \frac{(p-1)|S||T|}{p^2} \right| \leq \lambda \sqrt{|S||T|},$$

where $e(S,T)$ is the number of edges between $S$ and $T$.

$$e(S,T) \leq \frac{|S||T|}{p} + \sqrt{3p|S||T|}.$$

Set $S = (AA) \times (-A)$ and $T = (A^{-1}) \times (A + A)$.
There is an edge between any two vertices $(ab, -c) \in S$ and $(b^{-1}, a + c) \in T$, therefore the number of edges between $S$ and $T$ is at least $|A|^3/2$.

$$|A|^3 \leq e(S,T) \leq \frac{|S||T|}{p} + \sqrt{3p|S||T|} =$$

$$= \frac{|AA||A+A||A|^2}{p} + \sqrt{3p|AA||A+A||A|^2}.$$

After rearranging the inequality we get the desired sum-product bound.

$$|A+A||AA| \geq \min\left\{p|A|, \frac{|A|^4}{3p}\right\}.$$

In particular, if $|A| \approx p^{2/3}$, then $\max\{|AA|, |A+A|\} \gg |A|^{5/4}$.

# Integers

One can consider the natural numbers instead of finite fields,

$$V(G_{SP}) = \mathbb{N} \times \mathbb{N}.$$

Two vertices,

$$v_i = (a, b), v_j = (c, d) \in V(G_{SP}),$$

are connected by and edge,

$$(v_i, v_j) \in E(G_{SP}),$$

iff

$$ac = b + d.$$

Based on the previous arguments we are interested about the maximum possible number of edges between two vertex sets. Spectral bounds or similar discrepancy techniques won't work here.

Here is a closely related problem;
Among $n$ integers what is the maximum number of solutions for

$$xy = s + t?$$

The Elekes-type incidence bound works here – one can recover his $n^{5/4}$ bound – however much better bounds should hold here.

# The real case.

The best known bound is the following.

### Theorem

*Let A be a finite set of positive real numbers. Then*

$$|AA||A+A|^2 \geq \frac{|A|^4}{4\lceil \log |A| \rceil}.$$

The inequality is sharp - up to the power of the log term in the denominator - when $A$ is the set of the first $n$ natural numbers.

### Corollary

*Let A be a finite set of positive real numbers. Then*

$$\max\{|A+A|,|AA|\} \geq \frac{|A|^{4/3}}{2\lceil \log |A| \rceil^{1/3}}.$$

# Multiplicative Energy

We are going to use the notation of *multiplicative energy*. The name of this quantity comes from a paper of Tao, however its discrete version was used earlier.

Let $A$ be a finite set of reals. The multiplicative energy of $A$, denoted by $E(A)$, is given by

$$E(A) = |\{(a, b, c, d) \in A^4 | \quad \exists \lambda \in \mathbf{R} : (a, b) = (\lambda c, \lambda d)\}|.$$

In the notatation of Gowers, the quantity $E(A)$ counts the number of quadruples in $\log A$. (additive energy)

# Bounding the multiplicative energy

We will prove the following:

Let $A$ be a finite set of positive real numbers. Then

$$\frac{E(A)}{\lceil \log |A| \rceil} \leq 4|A+A|^2.$$

The main theorem then follows from via the Cauchy-Schwartz type inequality

$$E(A) \geq \frac{|A|^4}{|AA|}.$$

□

# Proof

Another way of counting $E(A)$ is the following:

$$E(A) = \sum_{x \in A/A} |xA \cap A|^2. \tag{1}$$

The summands on the right hand side can be partitioned into $\lceil \log |A| \rceil$ classes according to the size of $xA \cap A$.

$$E(A) = \sum_{i=0}^{\lceil \log |A| \rceil} \sum_{\substack{x \\ 2^i \leq |xA \cap A| < 2^{i+1}}} |xA \cap A|^2$$

There is an index, $I$, that

$$\frac{E(A)}{\lceil \log |A| \rceil} \leq \sum_{\substack{x \\ 2^I \leq |xA \cap A| < 2^{I+1}}} |xA \cap A|^2$$

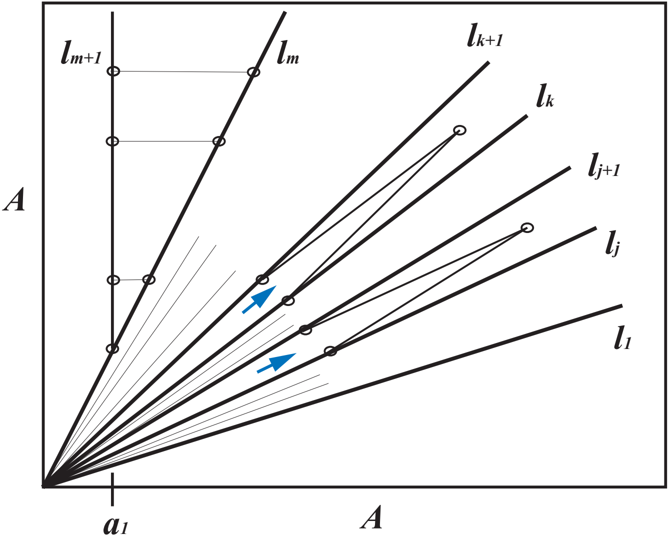
# Proof contd.

Let $D = \{s : 2^l \leq |sA \cap A| < 2^{l+1}\}$, and let $s_1 < s_2 < \ldots < s_m$ denote the elements of $D$, labeled in increasing order.

$$\frac{E(A)}{\lceil \log |A| \rceil} \leq \sum_{\substack{x \\ 2^l \leq |xA \cap A| < 2^{l+1}}} |xA \cap A|^2 < m 2^{2l+2}. \qquad (2)$$

Each line $l_j : y = s_j x$, where $1 \leq j \leq m$, is incident to at least $2^l$ and less than $2^{l+1}$ points of $A \times A$.

# Proof contd.



# Proof contd.

The sums are elements of $(A+A)\times(A+A)$, so we have the following inequality.

$$m2^{2l} \leq \left|\bigcup_{i=1}^{m}(l_i \cap A \times A) + (l_{i+1} \cap A \times A)\right| \leq |A+A|^2.$$

The inequality above with inequality (2) proves the lemma.

□

# Remark

Let $A$ and $B$ be finite sets of reals. The multiplicative energy, $E(A,B)$, is given by

$$E(A,B) = |\{(a,b,c,d) \in A\times B\times A\times B| \quad \exists\lambda \in \mathbf{R} : (a,b) = (\lambda c, \lambda d)\}|.$$

In the proof of the lemma we did not use the fact that $A = B$, the proof works for the asymmetric case as well. Suppose that $|A| \geq |B|$. With the lower bound on the multiplicative energy

$$E(A,B) \geq \frac{|A|^2|B|^2}{|AB|}$$

our proof gives the more general inequality

$$\frac{|A|^2|B|^2}{|AB|} \leq 4\lceil \log|B|\rceil |A+A||B+B|.$$

# More summands

We prove here a generalization of our bound for $k$-fold sumsets. For any integer $k \geq 2$ the $k$-fold subset of $A$, denoted by $kA$ is the set

$$kA = \{a_1 + a_2 + \ldots + a_k | a_1, \ldots, a_k \in A\}.$$

## Theorem

*For any integer k $\geq$ 2 there is a function $\delta = \delta_k(\varepsilon)$ that if $|AA| \leq |A|^{1+\varepsilon}$ then $|kA| \geq |A|^{2-1/k-\delta}$, where $\delta \to 0$ if $\varepsilon \to 0$.*

# Proof

We can suppose that $A$ has only positive elements WLOG.

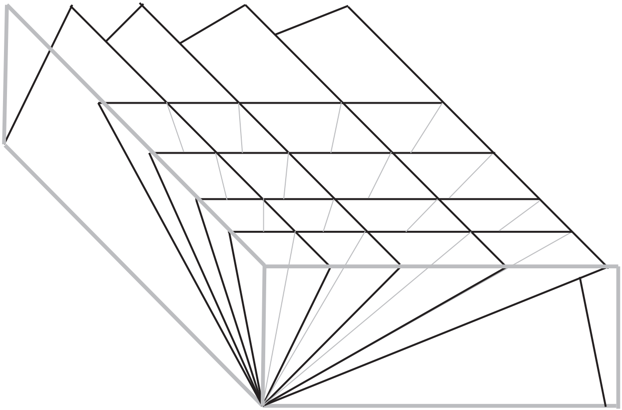
Let

$$|AA| \leq |A|^{1+\varepsilon}.$$

By a Plünnecke type inequality we have

$$|A/A| \leq |A|^{1+2\varepsilon}.$$

Consider the $k$-fold Cartesian product $A \times A \times \ldots \times A$, denoted by $\times^k A$. It can be covered by no more than $|A/A|^{k-1}$ lines going through the origin.



Let $H$ denotes the set of lines through the origin containing at least $|A|^{1-2\varepsilon(k-1)}/2$ points of $\times^k A$.

With this selection, the lines in $H$ cover at least half of the points in $\times^k A$, since

$$\frac{|A|^{1-2\varepsilon(k-1)}}{2}|A/A|^{k-1} = \frac{|A|^k}{2|A|^{(1+2\varepsilon)(k-1)}}|A/A|^{k-1} \leq \frac{|A|^k}{2}.$$

As no line has more than $|A|$ points common with $\times^k A$, therefore $|H| \geq |A|^{k-1}/2$.

The set of lines, $H$, represents a set of points, $P$, in the projective real space $\mathbf{RP}^{k-1}$. Point set $P$ has full dimension $k-1$ as it has a nice symmetry.

The symmetry follows from the Cartesian product structure; if a point with coordinates $(a_1, \ldots, a_k)$ is in $P$ then the point $(\sigma(a_1), \ldots, \sigma(a_k))$ is also in $P$ for any permutation $\sigma \in S_k$. Let us triangulate $P$.

By triangulation we mean a decomposition of the convex hull of $P$ into non-degenarate, $k-1$ dimensional, simplices such that the intersection of any two is the empty set or a face of both simplices and the vertex set of the triangulation is $P$.

Let $\tau(P)$ be a triangulation of $P$. We say that $k$ lines $l_1, \ldots, l_k \in H$ form a simplex if the corresponding points in $P$ are vertices of a simplex of the triangulation.

We use the following notation for this: $\{l_1, \ldots, l_k\} \in \tau(P)$. In the two-dimensional case we used that the sumsets of points on consecutive lines are disjoint.

Here we are using that the interiors of the simplices are disjoint, therefore sumsets of lines of simplices are also disjoint.

Note that we assumed that $A$ is positive, so we are considering convex combinations of vectors with positive coefficients. Let $\{l_1, \ldots, l_k\} \in \tau(P)$ and $\{l'_1, \ldots, l'_k\} \in \tau(P)$ are two distinct simplices. Then

$$\left(\sum_{i=1}^{k} l_i \cap \times^k A\right) \bigcap \left(\sum_{i=1}^{k} l'_i \cap \times^k A\right) = \emptyset.$$

Also, since the $k$ vectors parallel to the lines $\{l_1, \ldots, l_k\} \in \tau(P)$ are linearly independent, all sums are distinct,

$$\left|\sum_{i=1}^{k} l_i \cap \times^k A\right| = \prod_{i=1}^{k} \left|l_i \cap \times^k A\right|.$$

$$|kA|^k \geq \sum_{\{l_1,\dots,l_k\}\in\tau(P)} \left|\sum_{i=1}^{k} l_i \cap \times^k A\right| \geq \frac{|A|^{k-1}}{2k} \prod_{i=1}^{k} \left|l_i \cap \times^k A\right|.$$

Every line is is incident to at least $|A|^{1-2\varepsilon(k-1)}/2$ points of $\times^k A$, therefore

$$|kA|^k \geq \frac{|A|^{k-1+k(1-2\varepsilon(k-1))}}{2k2^k} = \frac{|A|^{2k-1-2k(k-1)\varepsilon}}{k2^{k+1}}.$$

Taking the $k$-th root of both sides we get the result we wanted to show

$$|kA| \geq c_k |A|^{2-1/k-2(k-1)\varepsilon}.$$