Applications and extensions of a theorem of Gallai

Gábor Simonyi
Rényi Institute, Budapest

In a seminal paper about comparability graphs Gallai proved the following results.

Theorem (Gallai 1967): If the edges of the complete graph K_n are colored so that no triangle gets 3 different colors, then at most 2 colors span a connected graph on the n vertices.

To honour this result we call an edge-coloring a Gallai-coloring if no triangle gets 3 different colors.

The above Theorem easily implies the following statement.

Corollary (Gallai 1967): A Gallai-colored complete graph can always be obtained from a 2-edge-colored complete graph by substituting Gallai-colored complete graphs into its vertices.

Statements expressible in terms of Gallai-colorings

Thm. (Gallai 1967): If a complete graph is Gallai-colored with $\mathbf{3}$ colors and each of two colors span comparability graphs then so does their union.

Inspired by this result the following theorem was proved.

Thm. (K. Cameron, Edmonds, Lovász 1986): If a complete graph is Gallai-colored and all but one of the colors give perfect graphs, then so does the last color (or equivalently, by the Perfect Graph Theorem: their union).

Thm. (Erdős, Simonovits, T. Sós 1973): If K_n is Gallai-colored, then at most $n-1$ colors can be used.

This result was originally stated in different terms and had a very elegant proof that was independent of Gallai's theorem.

(Assume for contradiction that at least n colors appear. Select an edge of each color. You have at least n edges on n vertices, so there is a cycle which is totally multicolored. Diagonals give shorter multicolored cycles, finally a multicolored triangle.)

But one can also easily prove the statement by using Gallai's theorem and induction on n.

Graph entropy

Körner defined graph entropy in 1971 as the solution of a problem in information theory. A later found short definition is this:

$$H(G,P) := \min_{(a_1,\ldots,a_n)\in VP(G)} \sum_{i=1}^{n} p_i \log \frac{1}{a_i},$$

where G is a graph on n vertices, $P = (p_1, \ldots, p_n)$ is a probability distribution on its vertex set, and $VP(G)$ is the vertex packing polytope of G.

The vertex packing polytope is the set of points in $\mathbb{R}^n$ that can be expressed as convex combinations of characteristic vectors of stable sets.

In particular,

$$H(K_n, P) = \min_{(q_1,\ldots,q_n)\in VP(K_n)} \sum_{i=1}^{n} p_i \log \frac{1}{q_i} =$$

$$\sum_{i=1}^{n} p_i \log \frac{1}{p_i} = H(P),$$

the Shannon entropy of the probability distribution P.

Subadditivity and additivity

An important property of $H(G,P)$ is its sub-additivity:

$H(F \cup G, P) \leq H(F,P) + H(G,P)$, where $V(F) = V(G) = V(F \cup G)$ and $E(F \cup G) = E(F) \cup E(G)$.

Thm. (Csiszár, Körner, Lovász, Marton, S. 1990)

$$\forall P : \quad H(G,P) + H(\overline{G},P) = H(P)$$

iff G is perfect.

Q: What about equality for general (not necessarily complementary) $\boldsymbol{F}$ and $\boldsymbol{G}$?

Thm. (Körner, S., Tuza 1992): Let F and G be two edge-disjoint graphs on the same vertex set V. We have

$$\forall P: \quad H(F \cup G, P) = H(F, P) + H(G, P)$$

iff the following two conditions are satisfied:

1. If on some $U \subseteq V$ all edges belong to $F \cup G$, then the induced subgraphs $F[U]$ and $G[U]$ are perfect.

2. Considering the edges of F, G and the rest as a 3-edge coloring, we get a Gallai-coloring.

Thm. (Körner, S. 2000): Let F and G be two graphs with possibly intersecting edge sets on the same vertex set V. Assume that $F \cup G = K_{|V|}$. Then

$$\forall P : H(F \cup G, P) + H(F \cap G, P) \leq H(F, P) + H(G, P)$$

holds iff coloring the edges according to their membership in $F - G$, $G - F$, and $F \cap G$ gives a Gallai-coloring.

Remark: The proof of both of the last two theorems rely on Gallai's theorem, that makes an inductive proof possible and very natural.

Gallai-colorings generalize 2-colorings

Thm. (Burr, unpublished): Any $\mathbf{2}$-coloring of a complete graph contains a monochromatic spanning broom.

(Broom: a path with a star on one of its endpoints.)

The Gallai-colored version is the following:

Thm. (Gyárfás, S. 2004): A Gallai-colored complete graph always contains a monochromatic spanning broom.

The proof uses Burr's ideas and Gallai's theorem.

Thm. (Erdős, Fowler 1999): A $\mathbf{2}$-edge-colored complete graph on $\boldsymbol{n}$ vertices always contains a monochromatic subgraph of diameter at most $\mathbf{2}$ with at least $\lceil \mathbf{3}\boldsymbol{n}/\mathbf{4} \rceil$ vertices.

Thm. (Gyárfás, G. Sárközy, Sebő, Selkow 2010): A Gallai-colored complete graph on $\boldsymbol{n}$ vertices always contains a monochromatic subgraph of diameter at most $\mathbf{2}$ with at least $\lceil \mathbf{3}\boldsymbol{n}/\mathbf{4} \rceil$ vertices.

The GySSS paper cited above gives conditions that make it automatic to obtain the Gallai-colored version of a theorem for $\mathbf{2}$-colorings. These cover several cases but not all of them.

Gallai-colorings of non-complete graphs

It is natural to start exploring properties of Gallai-colored graphs that are not necessarily complete, but are edge-colored so, that no tricolored triangle occurs.

Such investigations were initiated by Gyárfás and G. Sárközy, who proved the following result.

Thm. (Gyárfás, G. Sárközy 2010): Let G be a Gallai-colored graph with independence number α. Then G contains a monochromatic component of size at least $\frac{|V(G)|}{\alpha^2+\alpha-1}$.

For $G = K_n$ we get that 1 color class spans the whole vertex set. For 2-colorings this is a well-known and easy exercise.

Another result in this direction is the following.

Thm. (Gyárfás, S., Á. Tóth 2012): Let G be a Gallai-colored graph with independence number α. Then there is a function $g : \mathbb{N} \to \mathbb{N}$ such that the vertices of G can be covered by $g(\alpha)$ monochromatic components. If $\alpha = 2$ then at most 5 components are enough.

Note that for $\alpha = 2$ the last statement generalizes (the $\alpha = 2$ special case of) the previous theorem.

A further generalization is the following.

Thm. (Fujita, Furuya, Gyárfás, Á. Tóth 2012): Let G be a Gallai-colored graph with independence number α. Then there is a function $g : \mathbb{N} \longrightarrow \mathbb{N}$ such that the vertices of G can be partitioned into $g(\alpha)$ monochromatic components.

The point is that just eliminating the intersections in a covering (that exists by the previous theorem) may disconnect some of the so far monochromatic components. So this last theorem is stronger than the previous one.

Gallai-Ramsey numbers

Let $\mathrm{RG}(r, H)$ denote the minimum m such that in every Gallai-colored K_m with r colors, a monochromatic copy of H occurs.

A sample theorem:

Thm. (Gyárfás, G. Sárközy, Sebő, Selkow 2010):

For fixed H, $\mathrm{RG}(r, H)$ is exponential in r if H is not bipartite; linear in r if H is bipartite but not a star; and constant (independent of r) if H is a star.

Open problems

One of the most interesting open problems seems to be this:

Give necessary an sufficient conditions for a true statement about $\mathbf{2}$-edge-colored complete graphs to be "automatically" true for Gallai-colored complete graphs.

Another problem is the missing cases of properties of graph entropy. In particular, when is it true for all prob. dist. P that

$$H(F \cup G, P) + H(F \cap G, P) \leq H(F, P) + H(G, P)?$$