Local time of random walks on trees

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Simple random walk on $\mathbb{Z}$

$(X_n,\ n \ge 0)$ = simple random walk on $\mathbb{Z}$ with $X_0 = 0$.

Local time $L_n(x) := \#\{i : 0 \le i \le n,\ X_i = x\}$.

Set of favorite sites $A_n := \{x \in \mathbb{Z} :\ L_n(x) = \max_{y\in\mathbb{Z}} L_n(y)\}$.

Conjecture 1 (Erdős & Révész 1984):

almost surely, $0 \in A_n$ for infinitely many n.

Conjecture 2 (Erdős & Révész 1984):

a.s., $\#A_n \le 2$ for all sufficiently large n.

[Conjecture 2: a.s., $\#A_n \le 2$ for all sufficiently large n.]

Tóth (2001): a.s., $\#A_n \le 3$ for all sufficiently large n.

[Conjecture 1: a.s., $0 \in A_n$ for infinitely many n.]

Bass & Griffin (1986): a.s.,

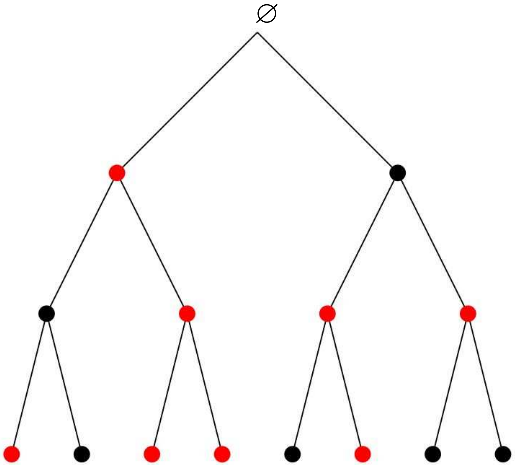
$$\liminf_{n\to\infty} \frac{(\log n)^a}{n^{1/2}} \min_{x\in A_n} |x| \text{ is } 0 \text{ if } a < 1, \text{ and is } \infty \text{ if } a > 11.$$

In particular, $\min_{x\in A_n} |x| \to \infty$ a.s.

Bass (2013+): a.s.,

$$\liminf_{n\to\infty} \frac{(\log n)^a}{n^{1/2}} \min_{x\in A_n} |x| \text{ is } 0 \text{ if } a \le 1, \text{ and is } \infty \text{ otherwise.}$$

Random walks on trees



$\mathbb{P}(\bullet) = 0.6$

$\mathbb{P}(\bullet) = 0.4$

Random walks on trees

$\mathbb{P}(\text{●}) = 0.6$ (red)

$\mathbb{P}(\text{●}) = 0.4$ (black)

Assumption: (wrong numerical values!)

$$\Big(\frac{0.3}{0.2} + \frac{0.5}{0.2}\Big) \times 0.6 + \Big(\frac{0.2}{0.7} + \frac{0.1}{0.7}\Big) \times 0.4 = 1,$$

$$\Big(\frac{0.3}{0.2} \ln \frac{0.3}{0.2} + \frac{0.5}{0.2} \ln \frac{0.5}{0.2}\Big) \times 0.6 + \Big(\frac{0.2}{0.7} \ln \frac{0.2}{0.7} + \frac{0.1}{0.7} \ln \frac{0.1}{0.7}\Big) \times 0.4 = 0.$$

(Useless) remark:

Our walk = random walk in random environment.

Our assumption says :

$$\mathbf{E}[\sum_{x:\,|x|=1} \mathrm{e}^{-V(x)}] = 1, \; \mathbf{E}[\sum_{x:\,|x|=1} V(x)\mathrm{e}^{-V(x)}] = 0.$$

V = potential associated with the random environment.

Under our assumption,

Lyons & Pemantle (1992): the walk is recurrent.

Faraud, Hu & S. (2012): the walk is very slow.

$$\frac{1}{(\log n)^3} \max_{k:\, 0\le k\le n} |X_k| \to \mathrm{const} \in (0,\, \infty), \text{ a.s.}$$

Open problem: prove that $\frac{|X_n|}{(\log n)^2}$ converges (at least) weakly.

Open problem: prove/disprove the Erdős-Révész conjecture 1.

Open problem: asymptotics of $\max_{x:\, x\in\mathbb{T}} L_n(x)$?

$$L_n(\varnothing) := \#\{i: \ 0 \le i \le n, \ X_i = \varnothing\}.$$

Theorem: $\frac{L_n(\varnothing)}{n/\ln n} \to \mathscr{L}$, in probability.

Remark. $0 < \mathscr{L} < \infty$ a.s. More precisely,

$$\mathscr{L} := \frac{1}{2}\frac{\sigma^2}{D_\infty},$$

$$\sigma^2 := \mathbf{E}[\textstyle\sum_{|x|=1} V(x)^2 \mathrm{e}^{-V(x)}] \in (0, \infty),$$

$D_\infty > 0$, $\quad =$ a.s. limit of the "derivative martingale".

$$\mathbf{E}\Big(\sum_{|x|=1} \mathrm{e}^{-V(x)} \Big) = 1, \qquad \mathbf{E}\Big(\sum_{|x|=1} V(x) \mathrm{e}^{-V(x)} \Big) = 0.$$

Proof: (Relatively) elementary, via

- spinal decompositions for branching random walk,
- study of excursions. □

Explanation of the limit $\mathscr{L} = \frac{1}{2} \frac{\sigma^2}{D_\infty}$:

- σ^2 and D_∞: from spinal decompositions.
- $\frac{1}{2}$: equals $\frac{1}{2} \sqrt{\frac{\pi}{2}} \, \frac{1}{c_{\mathfrak{m}}}$,

$$c_{\mathfrak{m}} := \mathbf{E}\big(\tfrac{1}{\eta}\big), \qquad \eta := \sup_{s \in [0,1]} (\overline{\mathfrak{m}}_s - \mathfrak{m}_s),$$

$$\overline{\mathfrak{m}}_s := \sup_{u \in [0,s]} \mathfrak{m}_u,$$

$$\mathfrak{m} = \text{Brownian meander}$$

$$= \text{“}\, (B_s,\ s \in [0,\ 1]) \,|\, B \geq 0 \text{ on } [0,\ 1] \,\text{”}.$$