On sum of powers of the positive integers

A. Schinzel
Institute of Mathematics
Polish Academy of Sciences
Warszawa

Problem

I regret to say that, as far as I know, Erdős has never considered the sums $1^k + 2^k + \ldots + n^k$, which are my subject today[1]. Theorem 2 of my talk is, however, similar to the theorem Erdős proved in his paper *On integers of the form $2^k + p$ and some related problems* in 1950.
W. Bednarek asked in a letter for a characterization of pairs of positive integers (k, m) such that for every positive integer n

$$1^k + 2^k + \ldots + n^k \mid 1^{km} + 2^{km} + \ldots + n^{km}. \qquad (1)$$

[1] The first sentence is not quite true, since Erdős and Moser formulated a conjecture about the Diophantine equation $1^k + 2^k + \ldots + (x-1)^k = x^k$ (added during the conference).

Partial answer

The following theorem contains a partial answer with the help of Bernoulli numbers. They are denoted by B_n:

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ldots,\ B_{2l+1} = 0,$$

and the Bernoulli polynomial $\sum_{l=0}^{n}\binom{n}{l}B_l x^{n-l}$ by $B_n(x)$.

Theorem 1

Theorem 1. *If the divisibility* (1) *holds for every positive integer n, then m is odd and*

$$\begin{aligned} B_{km}/B_k &\in \mathbb{Z} \text{ for } k \text{ even}, \\ mB_{km-1}/B_{k-1} &\in \mathbb{Z} \text{ for } k \text{ odd} \geq 3. \end{aligned} \tag{2}$$

The condition is sufficient for $k \leq 3$, but insufficient for $k = 4$ and infinitely many m.

Conjecture

In fact we propose

Conjecture. *For* $k > 3$ *the divisibility* (1) *holds for every positive integer* n *only for* $m = 1$.

To support this conjecture we have

Theorem 2. *For* $k = 4$, $n \equiv 58966743 \pmod{5^6 \cdot 11251^2}$ *the divisibility* (1) *holds only for* $m = 1$.

Theorem 3. *For* $m = n = 3$ *the divisibility* (1) *holds only for* $k \leq 3$.

Proof of Theorem 1

The proof of Theorem 1 is based on five lemmas.

Lemma 1. *For all positive integers k and n*

$$1^k + \ldots + (n-1)^k = S_k(n) = \frac{1}{k+1}(B_{k+1}(n) - B_{k+1}).$$

This is classical.

Lemma 2. *If $P, Q \in \mathbb{Q}[x]$ and $P(n)/Q(n) \in \mathbb{Z}$ for all sufficiently large integers n then $r(x) = P(x)/Q(x)$ is an integer-valued polynomial.*

This is easy.

Proof of Theorem 1

Lemma 3. *If* $3^\nu \parallel 2N$, *where* $N = n, n+1$ *or* $n + \frac{1}{2}$ *and* $\nu \geq 1$, *then for every positive integer* m

$$3^{\nu-1} \mid S_{2m}(n+1).$$

Lemma 4. *If* $2^\nu \parallel N$, *where* $N = n$ *or* $n+1$ *and* $\nu \geq 1$, *then for every positive integer* $r > 2$

$$2^{\nu-1} \mid S_{2r}(n+1).$$

Proofs of both lemmas are tedious.

Proof of Theorem 1

Lemma 5. *If a prime p satisfies $p - 1 \nmid k$, then p does not divide the denominator of B_k. If $p - 1 \mid k$, then p occurs in the denominator of B_k in the first power only.*

This is the von Staudt theorem.

Proof of Theorem 1

Proof of Theorem 1. Necessity. Since (1) holds for $n = 2$ we obtain $m \equiv 1 \pmod 2$. Consider now k even. By Lemma 1 we have

$$S_k(n) = \frac{1}{k+1} B_{k+1}(n), \quad S_{km}(n) = \frac{1}{km+1} B_{km+1}(n),$$

hence, for all integers $n > 1$, $B_{k+1}(n) > 0$ and

$$\frac{k+1}{km+1} \frac{B_{km+1}(n)}{B_{k+1}(n)} \in \mathbb{Z}.$$

Proof of Theorem 1

By Lemma 2

$$r(x) = \frac{k+1}{km+1}\,\frac{B_{km+1}(x)}{B_{k+1}(x)}$$

is an integer-valued polynomial and, since $r(0) = B_{km}/B_k$, (2) follows.

Consider next $k \geq 3$ odd. We have by Lemma 1

$$S_k(n) = \frac{1}{k+1}(B_{k+1}(n) - B_{k+1}),$$

$$S_{km}(n) = \frac{1}{km+1}(B_{km+1}(n) - B_{km+1}),$$

hence, for all integers $n > 1$, $B_{k+1}(n) > B_{k+1}$ and

$$\frac{k+1}{km+1}\,\frac{B_{km+1}(n) - B_{km+1}}{B_{k+1}(n) - B_{k+1}} \in \mathbb{Z}.$$

Proof of Theorem 1

By Lemma 2

$$r(x) = \frac{k+1}{km+1} \frac{B_{km+1}(x) - B_{km+1}}{B_{k+1}(x) - B_{k+1}}$$

is an integer-valued polynomial and, since $r(0) = mB_{km-1}/B_{k-1}$, (2) follows.

Proof of sufficiency for $k \leq 3$ is tedious.

Proof of Theorem 1

Insufficiency for $k = 4$. Take m to be a prime $\equiv 17 \pmod{30}$. The condition (2) is fulfilled, since $B_{4m}/B_4 = -30B_{4m} \in \mathbb{Z}$. Indeed, by Lemma 5, B_{4m} has in the denominator only the first powers of primes p such that $p - 1 \mid 4m$. The divisibility gives $p = 2, 3, 5, 2m + 1$ or $4m + 1$. Now, $2 \cdot 3 \cdot 5 = 30$, $2m + 1$ is divisible by 5 and $4m + 1$ by 3. It follows from Theorem 2 that $S_4(n + 1) \nmid S_{4m}(n + 1)$ for a positive integer n.

Lemmas to Theorem 2

The proof of Theorem 2 is based on four lemmas.

Lemma 6. *If p is a prime, $k' \equiv k \not\equiv 0 \pmod{p-1}$ and $n' \equiv n \pmod{p}$, then*

$$S_{k'}(n') \equiv S_k(n) \pmod{p}.$$

Lemma 7. *If $p > 2$ is a prime, $k \geq \alpha \geq 2$, $k' \geq \alpha$, $k \not\equiv 0 \pmod{p(p-1)}$, $k' \equiv k \pmod{p^{\alpha-1}(p-1)}$ and $n' \equiv n \pmod{p^{\alpha+1}}$, then*

$$S_{k'}(n') \equiv S_k(n) \pmod{p^\alpha}.$$

Lemma 8

Lemma 8. *If* $n \equiv 58966743 \pmod{11251^2}$*, then* $S_{4m}(n+1) \equiv 0 \pmod{11251}$ *only if* $m \equiv 1 \pmod{5625}$.

Proof. The number $p = 11251$ is a prime and $n \equiv 252 \pmod{p}$, $\left\lfloor \frac{n}{p} \right\rfloor \equiv 5241 \pmod{p}$. If $4m \equiv 0 \pmod{p-1}$, then

$$S_{4m}(n+1) \equiv n - \left\lfloor \frac{n}{p} \right\rfloor \equiv -4989 \not\equiv 0 \pmod{p}.$$

If $4m \not\equiv 0 \pmod{p-1}$, it suffices by Lemma 6 to verify the congruence $S_{4m}(252) \equiv 0 \pmod{p}$ for m in the interval $[1, 11249]$. The verification has been performed by J. Browkin.

Lemma 9

Lemma 9. *If* $n \equiv 58966743 \pmod{5^6}$, *then* $S_{4m}(n+1) \equiv 0 \pmod{5^5}$ *only if* $m = 1$ *or* $m \equiv 501 \pmod{625}$.

Proof. We have $58966743 \equiv 13618 \pmod{5^6}$. If $m \equiv 0 \pmod 5$, then

$$S_{4m}(n+1) \equiv n - \left\lfloor \frac{n}{5} \right\rfloor \equiv 13618 - 2723 = 10895 \not\equiv 0 \pmod{25}.$$

If $m \not\equiv 0 \pmod 5$, it suffices by Lemma 7 to verify the congruence $S_{4m}(13619) \equiv 0 \pmod{5^5}$ for m in the interval $[1, 626]$. The verification has been performed by J. Browkin.

Proof of Theorem 2

Proof of Theorem 2. Since for $n \equiv 58966743 \pmod{5^6 \cdot 11251^2}$ we have

$$S_4(n+1) \equiv 0 \pmod{5^5 \cdot 11251}$$

the theorem follows from Lemmas 8 and 9.

Proof of Theorem 3

The proof of Theorem 3 is based on

Lemma 10. *For every positive integer k*

$$(2^k, 1 + 2^k + 3^k) \leq 4,$$
$$(3^{k+1}, 1 + 2^k + 3^k) \leq 3k.$$

This is easy.

Proof of Theorem 3

Proof of Theorem 3. We have

$$1+2^{3k}+3^{3k}-2^k\cdot 3^{k+1} = (1+2^k+3^k)(1+2^{2k}+3^{2k}-2^k-3^k-6^k),$$

thus if (1) holds, then

$$1+2^k+3^k \mid 2^k\cdot 3^{k+1}. \qquad (3)$$

By Lemma 10 $(2^k\cdot 3^{k+1}, 1+2^k+3^k)\leq 12k$, thus by (3)

$$1+2^k+3^k\leq 12k,$$

which implies $k\leq 3$.