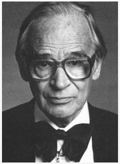
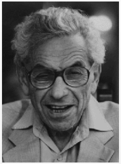
Extremal combinatorics in random discrete structures

Mathias Schacht

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Erdős Centennial Conference

Ramsey Theory

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- discrete structure G with Ramsey property and additional properties
  - "smallest" G
  - G with similar properties as F itself

Ramsey theory – classical results

Example from Hamburg

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Theorem (van der Waerden '27)

$\forall k$, $\forall r$, $\exists n_0 = n_0(k, r)$, such that for every partition

$$[n] = \{1, \ldots, n\} = C_1 \dot{\cup} \ldots \dot{\cup} C_r, \quad n \geq n_0,$$

there exists a class C_i, which contains an arithmetic progression of length k (AP_k). I.e. for sufficiently large $n \geq n_0(k, r)$ we have

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Theorem (Ramsey '30)

$$\forall F, r\ \exists n_0\ \forall n \geq n_0 : \qquad K_n^{(k)} \longrightarrow (F)_r^e\,.$$

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Turán Theory

For a k-uniform hypergraph F and $n \in \mathbb{N}$ set

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$\forall$ *graph* ($k = 2$) F

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- π_F known only for very few hypergraphs $k \geq 3$

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Conjecture (Erdős-Turán '36)

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- Extension of van der Waerden's theorem
- multidimensional and polynomial extensions known
- density version of the Hales-Jewett theorem

Relative Versions

Question

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- $X = \{2, 3, 5, 7, 11, \dots\}$ Green-Tao

Sum-free Sets

Observation

If $A \subseteq [n]$ with

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then there exist $x, y, z \in A$ such that $x + y = z$.

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- Ramsey version due to Schur '17: $[r!e] \longrightarrow (x + y = z)_r$.

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$\forall\delta > 0$, $\exists\xi > 0$ and n_0 such that $\forall n \geq n_0$ we have If $U \subseteq V_n$ and

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Main Question

What are the asymtotics of the smallest sequence $(p_n)_{n\in\mathbb{N}}$ of probabilities such that α-density from $(H_n)_{n\in\mathbb{N}}$ can be transferred to $(H_n[V_{n,p_n}])_{n\in\mathbb{N}}$?

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Second Idea

A.a.s. we need

$$e(H_n[V_{n,p_n}]) \gg |V_{n,p_n}| .$$

Examples:

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- $p^{e(F')} n^{v(F')} \gg pn^k \quad \forall F' \subseteq F$ — Turán

Extremal Combinatorics in Random Sets

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$\forall k \geq 3$, $\forall \delta > 0$, $\exists 0 < c < C$, such that $\forall (q_n)_{n \in \mathbb{N}}$

$$\lim_{n \to \infty} \mathbb{P}\big(r_k([n]_{q_n}) \leq \delta q_n n\big) = \begin{cases} 1, & \text{if } q_n \geq Cn^{-1/(k-1)}, \\ 0, & \text{if } q_n \leq cn^{-1/(k-1)}. \end{cases}$$

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- Main result yields probabilistic versions of many extremal results
  - multidimensional and polynomial variants of Szemerédi's theorem
  - maximal sum-free subsets
  - theorems of Turán and of Erdős and Stone for $G(n, p)$ and $G^{(k)}(n, p)$

Remarks

- probabilistic version of Turán's theorem was conjectured by Kohayakawa, Łuczak, and Rödl and only known for a few graphs F: K_3, K_4, K_5, K_6, trees, cycles (KŁR, Haxell, Steger et al.)

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- new proofs for more general results were found recently by:
  - Balogh, Morris and Samotij
  - Saxton and Thomason
- joint work with Conlon, Gowers and Samotij shows that approaches give a "Counting Lemma" for Szemerédi's regularity lemma for subgraphs of $G(n, p)$
  $\rightarrow$ probabilistic version of the Removal Lemma

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