Oscillation and edge labelled graphs

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A metric space $\mathrm{M} = (M; \mathrm{d})$ has property OS if:

For every $\epsilon > 0$ and every unif. cont. bdd. function $f : M \to \Re$ there exists a copy $\mathrm{M}^* = (M^*; \mathrm{d})$ of M in M for which

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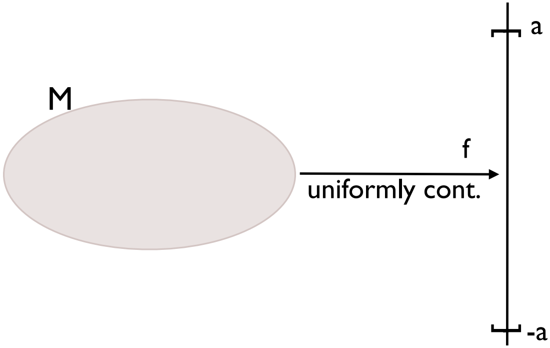

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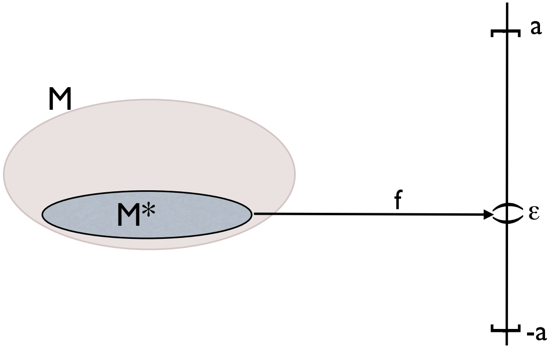

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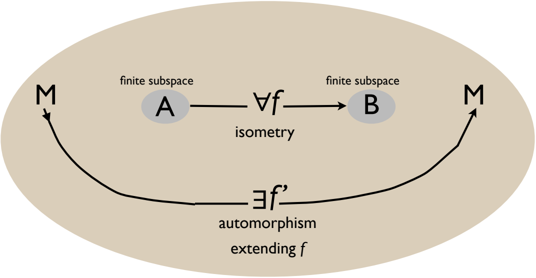

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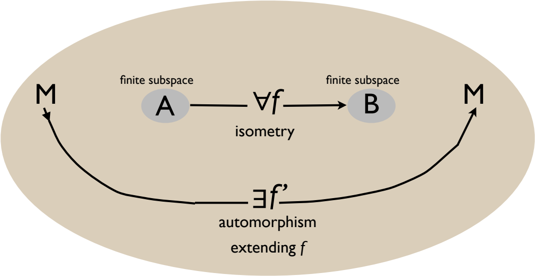

A metric space M is *universal* if it embeds every finite metric space F with dist $\mathrm{F} \subseteq$ dist M.

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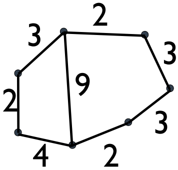

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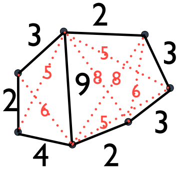

What are the conditions on a set $\mathcal{R}$ of non negative reals so that every “*metrizable*" edge labelled graph with labels in $\mathcal{R}$ can be embedded, label preserving, into a metric space with distances in $\mathcal{R}$?

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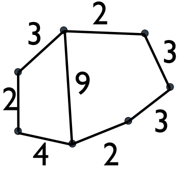

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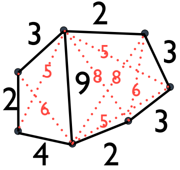

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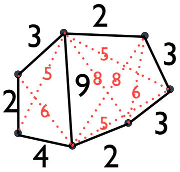

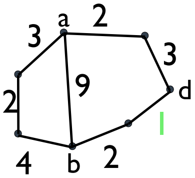

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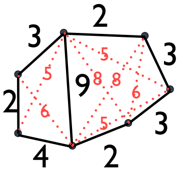

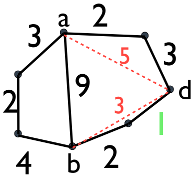

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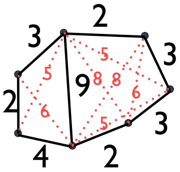

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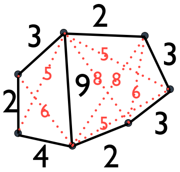

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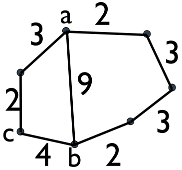

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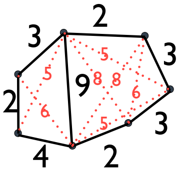

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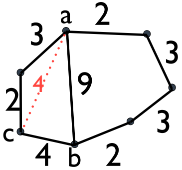

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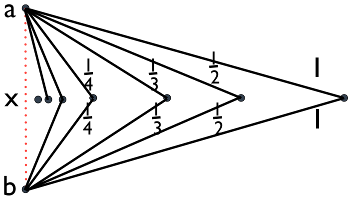

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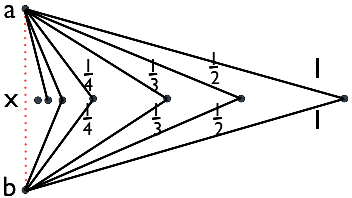

$x \leq \frac{2}{n}$ for all n, hence $x = 0$.

The distance x of a to b is “forced” to be 0.

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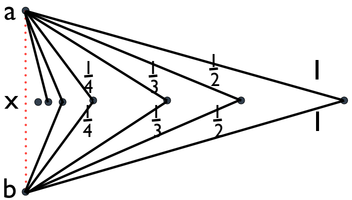

An edge labeled graph is *regular* if distances between different points are not “forced” to be 0.

Let 0 and r be a limit of the set of distances $\mathcal{R}$.

with $\lim s_n = 0$ and $\lim r_n = r$.

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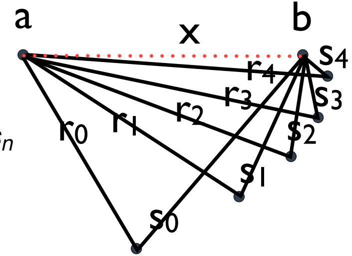

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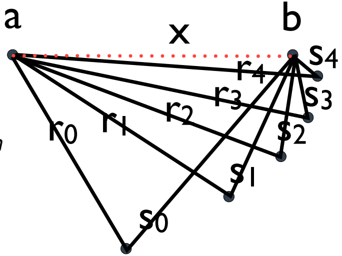

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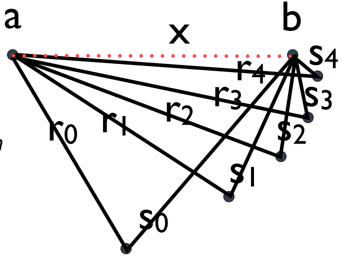

Then $|x - r| \leq |x - r_n| + |r_n - r| \leq s_n + |r_n - r| < \epsilon$. Hence $r \in \mathcal{R}$.

It follows that if 0 is a limit then one of the conditions on $\mathcal{R} \subseteq \Re_{\geq 0}$ has to be that $\mathcal{R}$ is a closed subset of the reals.

A metric space $\mathrm{M} = (M; \mathrm{d})$ has property OS if for every $\epsilon > 0$ and every uniformly continuous and bounded function $f : M \to \Re$ there exists a copy $\mathrm{M}^* = (M^*; \mathrm{d})$ of M in M for which:

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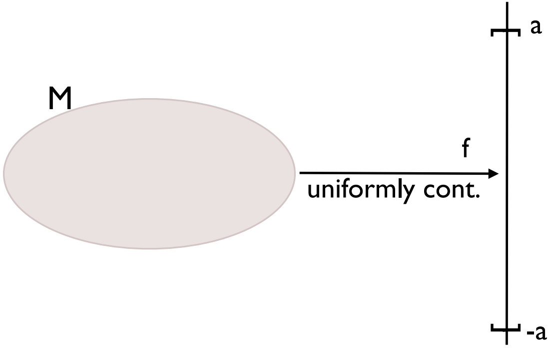

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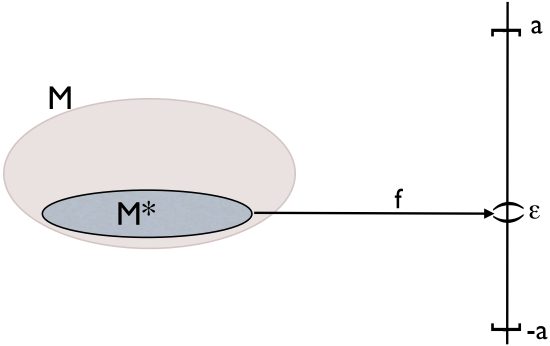

A metric space M is *approximately indivisible* if:

For every $\epsilon > 0$ and $n \in \omega$ and function $\gamma : M \to n$

there exist $i \in n$ and an isometric copy $\mathrm{M}^* = (M^*; \mathrm{d})$ of M with

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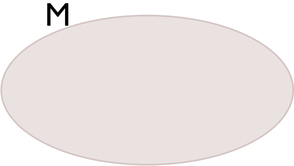

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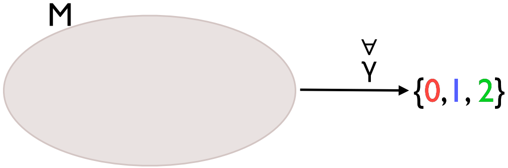

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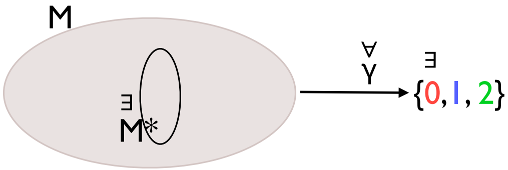

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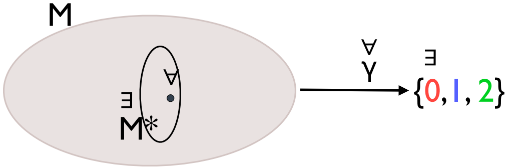

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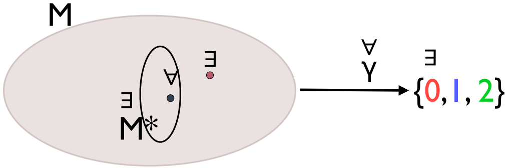

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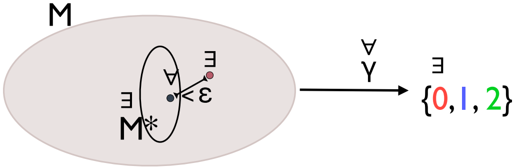

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Theorem (Kechris-Pestov-Todorcevic)

In the case of homogeneous metric spaces the properties of

approximately indivisible and OS

and oscillation stable are equivalent.

Let $\mathcal{R}$ be a closed subset of the reals $\Re_{\geq 0}$, then:

$$s \oplus_{\mathcal{R}} r := \sup\{x \in \mathcal{R} \mid x \leq s + r\}.$$

$\mathcal{R}$ is closed under $\oplus$ which is a commutative operation on $\mathcal{R}$.

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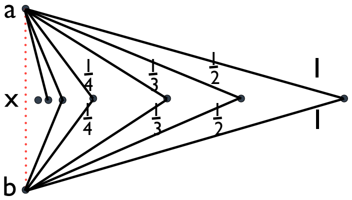

Definition

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An $\mathcal{R}$-edge labelled graph is metric *if it is regular and the $\oplus$-sum over the labels of a path is larger than or equal to the label of the edge connecting the endpoints of the path.*

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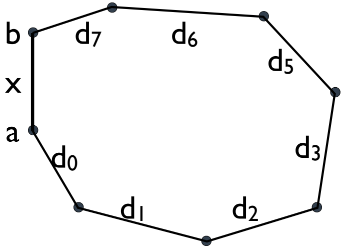
$$x \leq \bigoplus_{i \in 8} d_i$$



Theorem

If $\mathcal{R}$ is a closed subset of the reals then

every metric $\mathcal{R}$-edge labelled graph has an extension to a

metric space M with $\mathrm{dist}(\mathrm{M}) \subseteq \mathcal{R}$ iff $\oplus_{\mathcal{R}}$ is associative.

If 0 is a limit of $\mathcal{R}$ and every metric $\mathcal{R}$-edge labelled graph has

an extension to a metric space then

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Theorem

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There exists a

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iff $\mathcal{R}$ is a closed subset of the reals and $\oplus_{\mathcal{R}}$ is associative.

The space $\boldsymbol{U}_{\mathcal{R}}$ has property OS,

or equivalently it is approximately indivisible,

iff $\mathcal{R}$ is also bounded.

Theorem

If $\mathcal{R}$ is finite and $\oplus_{\mathcal{R}}$ is associative then:

there exists a countable homogeneous metric space, $\boldsymbol{U}_{\mathcal{R}}$,

which is indivisible.

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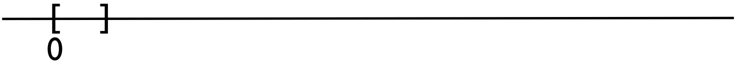

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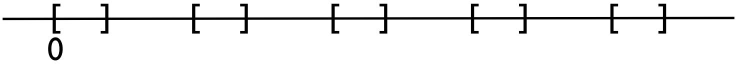

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There exists a finite subset $A \subseteq \mathcal{R}$ so that for every $l \in \mathcal{R}$:

$$\bar{l}^A := \min\{x \in A \mid x \geq l\} < \epsilon.$$

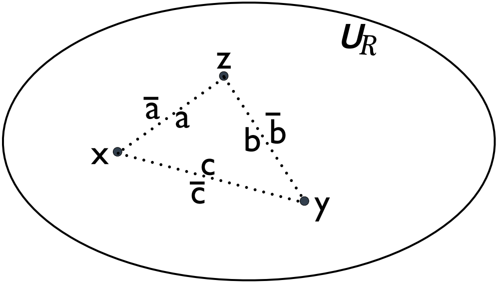
Theorem

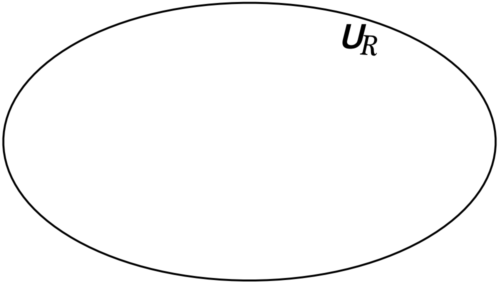
There exists a finite set B with $A \subseteq B \subseteq \mathcal{R}$ and $\oplus_B$ associative.

Lemma

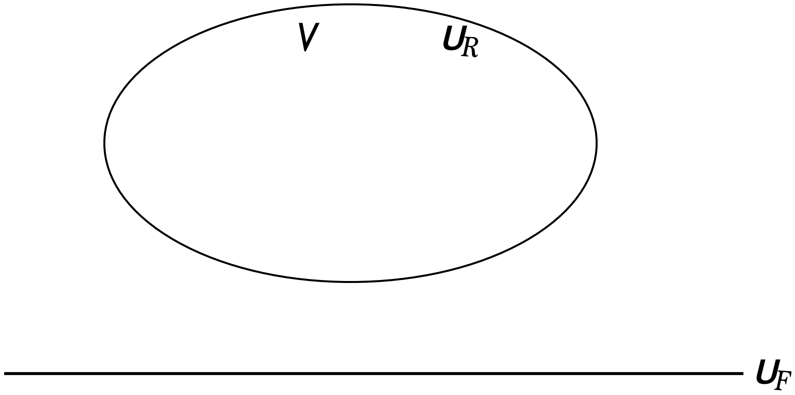
Let $\mathcal{R} \subseteq \Re$ be closed and bounded with 0 as a limit and $\oplus_{\mathcal{R}}$ associative. Let $A \subseteq \mathcal{R}$ be finite so that $\oplus_A$ is associative. ($\bar{x} - x < \epsilon$ *for all* $x \in \mathcal{R}$.)

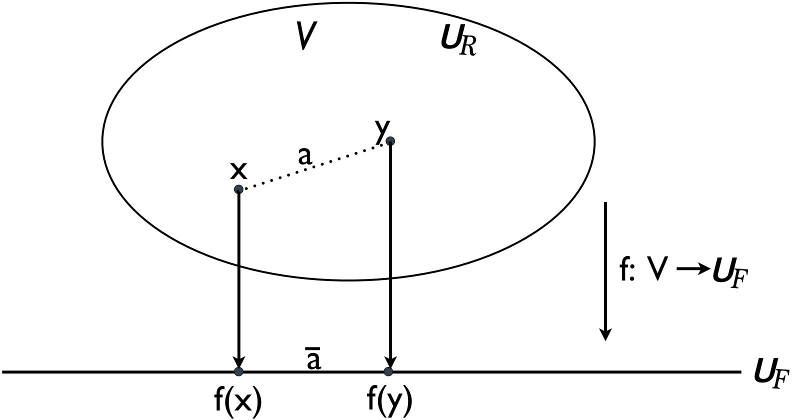
Then if $\{a, b, c\} \subseteq \mathcal{R}$ is metric the set $\{\bar{a}, \bar{b}, \bar{c}\}$ is metric.



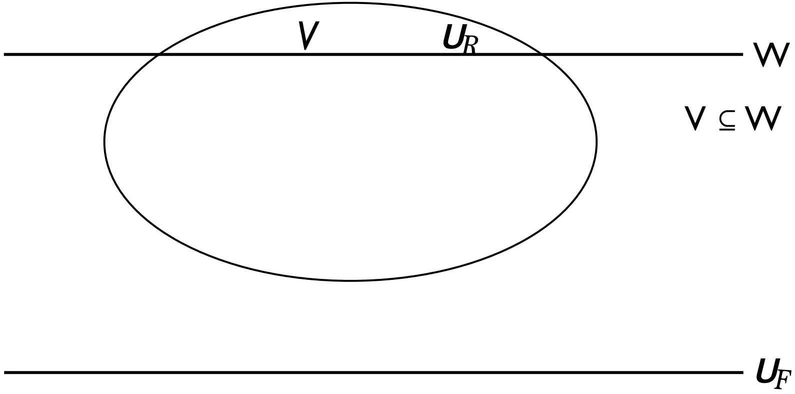


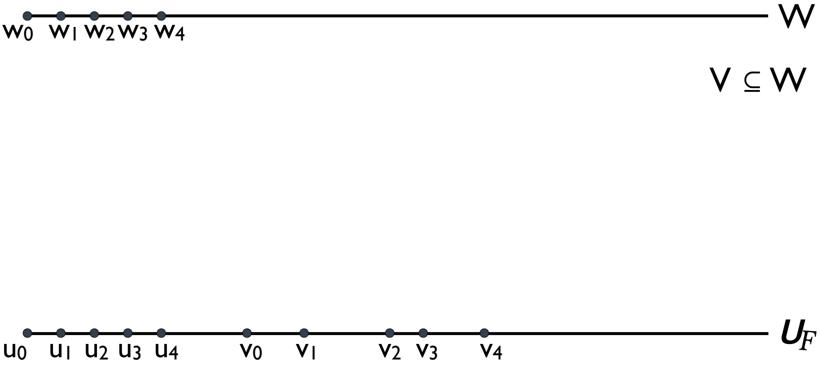
U_R


V
U_R
U_F

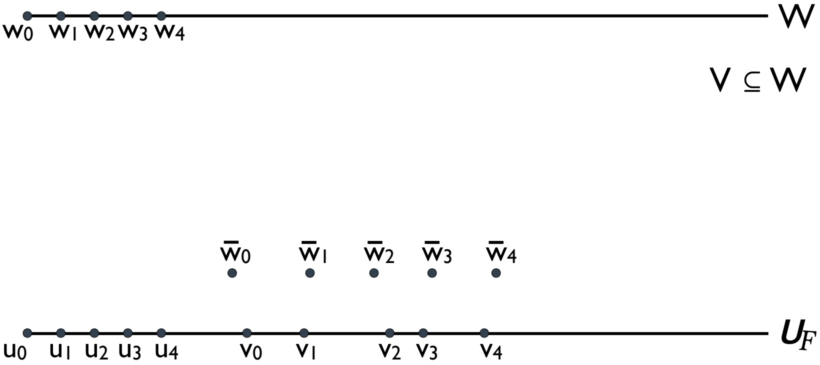


V
U_R
y
a
x
f: V → U_F
ā
U_F
f(x)
f(y)

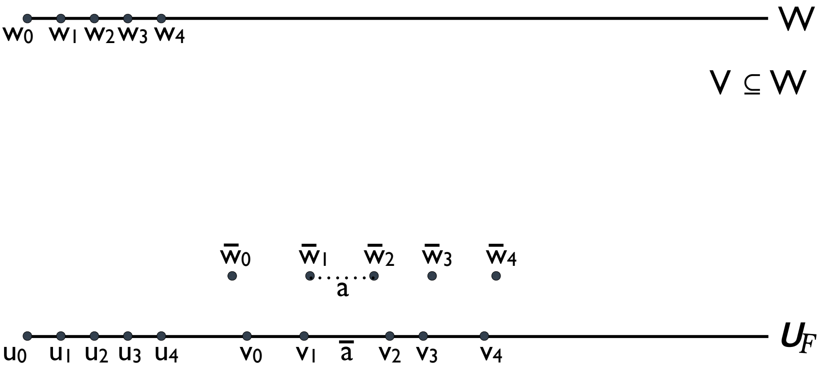

V
U_R
W
V ⊆ W
U_F



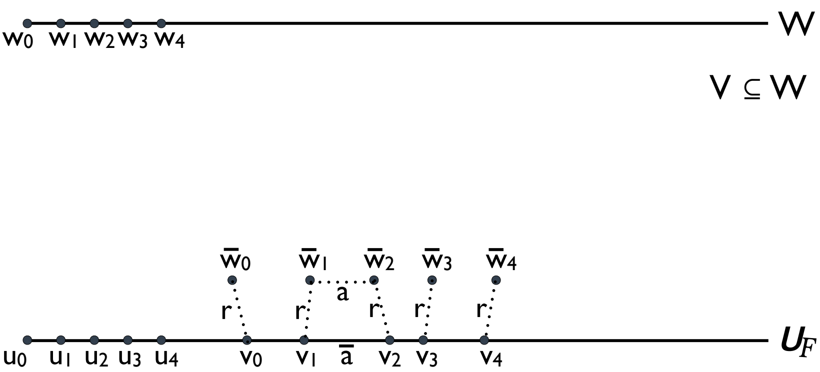
W
w0 w1 w2 w3 w4
V ⊆ W
U_F
u0 u1 u2 u3 u4
v0 v1 v2 v3 v4



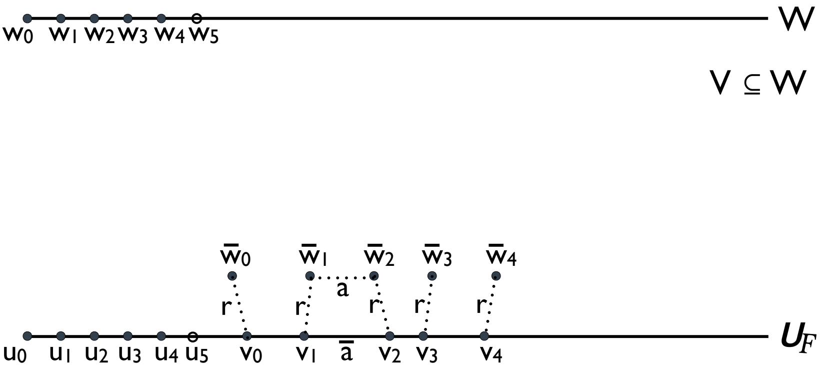
W
w_0 w_1 w_2 w_3 w_4
$V \subseteq W$
$\bar{w}_0$ $\bar{w}_1$ $\bar{w}_2$ $\bar{w}_3$ $\bar{w}_4$
U_F
u_0 u_1 u_2 u_3 u_4
v_0 v_1 v_2 v_3 v_4

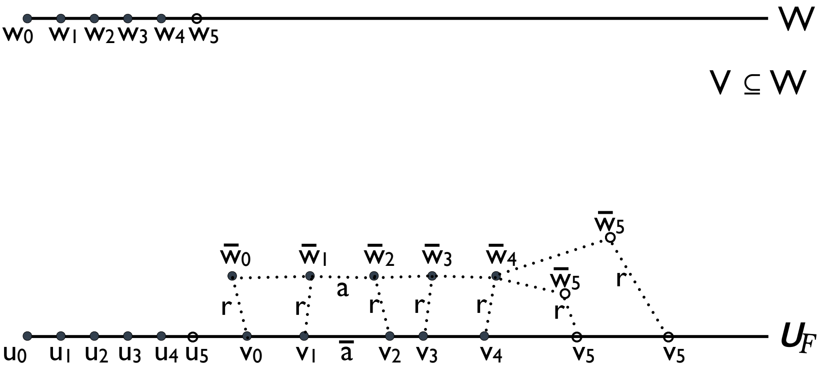


w0 w1 w2 w3 w4
W
V ⊆ W
w̄0 w̄1 w̄2 w̄3 w̄4
a
u0 u1 u2 u3 u4 v0 v1 ā v2 v3 v4
U_F



W
w0 w1 w2 w3 w4
V ⊆ W
w̄0 w̄1 w̄2 w̄3 w̄4
a
r r r r r
U_F
u0 u1 u2 u3 u4
v0 v1 ā v2 v3 v4


w0 w1 w2 w3 w4 w5
W
V ⊆ W
w̄0 w̄1 w̄2 w̄3 w̄4
a
r r r r r
u0 u1 u2 u3 u4 u5 v0 v1 ā v2 v3 v4
U_F



W
w0 w1 w2 w3 w4 w5
V ⊆ W
w̄5
w̄0 w̄1 w̄2 w̄3 w̄4
w̄5
a
r r r r r r r
U_F
u0 u1 u2 u3 u4 u5 v0 v1 ā v2 v3 v4 v5 v5

Lemma

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Finally $\bar{c} \leq \bar{a} \oplus \bar{b} \leq \bar{a} + \bar{b}$.

□