# ON ADDITIVE AND MULTIPLICATIVE DECOMPOSITIONS OF SUBSETS OF $\mathbb{F}_{p}$ 

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I visited him soon. I told him my results. We had a nice discussion and he asked a related question (which was sort of converse of the problem asked earlier by Turán). Roughly, this question was: is it true that every "dense" sequence of non-negative integers is reducible? If the answer is affirmative, then how dense a sequence must be to guarantee reducibility? As an answer to this question, I soon published (again in the Acta Arithmetica) my first paper based on an Erdős problem.

This was the first "Uncle Paul session" that I attended, and it was followed by many others. During one of the next sessions Erdős asked the following question: "It is easy to see that the sequence of the squares is totalprimitive. Is it also true that if we change this sequence so that we change \(o(\sqrt{n})\) elements up to \(n\) then the new sequence must be also totalprimitive?" Szemerédi and I settled this problem nearly completely, and we wrote a joint paper on must be also totalprimitive?" Szemerédi and I settled this problem nearly completely, and we wrote a joint paper
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In particular, it has been proved: if there are \(\mathcal{P}^{\prime} \sim \mathcal{P}\) and \(\mathcal{A}, \mathcal{B}\) with
\[
\mathcal{P}^{\prime}=\mathcal{A}+\mathcal{B}, \quad|\mathcal{A}|,|\mathcal{B}| \geq 2
\]
then we have
\[
\frac{n^{1 / 2}}{(\log n)^{c_{1}}}<A(n), B(n)<n^{1 / 2}(\log n)^{c_{2}} \quad\left(\text { for } n>n_{0}\right)
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where \(A(n), B(n)\) are the counting functions of \(\mathcal{A}\) and \(\mathcal{B}\), and \(c_{1}, c_{2}\) are positive absolute constants, and Elsholtz also proved:
if
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then there are no \(\mathcal{A}, \mathcal{B}, \mathcal{C}\) with
\[
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He also studied multiplicative decompositions of the set of the shifted primes, i.e., decompositions of the form
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Recently Shkredov and Shparlinski have improved independently on Theorem 1: they proved that it follows from the same assumptions that

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with some positive absolute constants $c_{1}<1, c_{2}>1$.
They used different approach: They used the fact that $\mathcal{Q}$ is a subgroup of the multiplicative group of $\mathbb{F}_{p}^{*}$. Shparlinski also proved similar results on additive 2-decompositions of other multiplicative subgroups $\mathcal{G}$ of $\mathbb{F}_{p}^{*}$.

While their methods use more special properties of the quadratic residues and thus they give sharper estimates, my method gives slightly weaker estimates but, on the other hand, it has the advantage that it also works in more general situations, e.g., it can be also used for studying additive properties of polynomial sets $\left\{f\left(x^{d}\right): x \in \mathbb{F}_{p}\right\}$ where $f$ is a permutation polynomial.

For a set $\mathcal{A}$ write $\mathcal{A} \hat{+} \mathcal{A}=\left\{a+a^{\prime}: a, a^{\prime} \in \mathcal{A}, a \neq a^{\prime}\right\}$. Shkredov also determined all the primes $p$ for which $Q=Q(p)$ has a special additive decomposition of the form $\mathcal{A}+\mathcal{A}$ or $\mathcal{A} \hat{+} \mathcal{A}$.

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While their methods use more special properties of the quadratic residues and thus they give sharper estimates, my method gives slightly weaker estimates but, on the other hand, it has the advantage that it also works in more general situations, e.g., it can be also used for studying additive properties of polynomial sets $\left\{f\left(x^{d}\right): x \in \mathbb{F}_{p}\right\}$ where $f$ is a permutation polynomial.

For a set $\mathcal{A}$ write $\mathcal{A} \hat{+} \mathcal{A}=\left\{a+a^{\prime}: \quad a, a^{\prime} \in \mathcal{A}, a \neq a^{\prime}\right\}$. Shkredov also determined all the primes $p$ for which $Q=Q(p)$ has a special additive decomposition of the form $\mathcal{A}+\mathcal{A}$ or $\mathcal{A} \hat{+} \mathcal{A}$

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## 3. On additive decompositions of the set of the primitive roots modulo $p$

In a joint paper (Monatshefte Math., 2013) with C. Dartyge we studied the set $\mathcal{G}(p)=\left\{g: g \in \mathbb{F}_{p,} g\right.$ is a primitive root modulo p\}. We conjectured:

Conjecture 3
If $p>p_{0}$ then $\mathcal{G}=\mathcal{G}(p)$ is primitive (i.e., it has no 2-decomposition).
Again, the conjecture seems to be beyond reach but we proved partial results similar to the results proved in case of the quadratic residues:

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where $\varphi(n)$ is Euler's function and $\tau(n)$ denotes the divisor function.

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From the last theorem we derived (using again Ruzsa's theorem):
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In Theorems 1 and 2 I studied additive 2- and 3-decompositions of the set \(\mathcal{Q}=\left\{x^{2}: x \in \mathbb{F}_{p}^{*}\right\}\). One might like to study the multiplicative analogues of these results by considering (non-trivial) 2- and 3-decompositions \(\mathcal{A} \cdot \mathcal{B}\), resp. \(\mathcal{A} \cdot \mathcal{B} \cdot \mathcal{C}\) with \(|\mathcal{A}|,|\mathcal{B}|,|\mathcal{C}| \geq 2\). However, some caution is needed:

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Next, observe that if \(\mathcal{A} \subset \mathbb{F}_{p}, 0 \in \mathcal{A}\) and \(|\mathcal{A}| \geq 2\), then
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If $c \neq 0$, and we write

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Again, this conjecture seems to be beyond reach, however, I proved partial results similar to Theorems 1 and 2 proved in the case of additive decompositions of $\mathcal{Q}$ :

Theorem 5
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#### Abstract

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## 5. On the reducibility of large subsets of $\mathbb{F}_{p}$

I mentioned my early papers answering the questions of Turán and Erdős on the reducibility of dense sets of non-negative integers. In a recent joint paper with K. Gyarmati and S. Konyagin (Journal of Number Theory, 2013) we studied the finite analogues of these old results of mine: we estimated the cardinality $f(p)$ of the largest primitive subset of $\mathbb{F}_{p}$.

Note that Green, Gowers and Green, and Alon studied a closely related problem: they studied representations of large subsets $\mathcal{C}$ of $\mathbb{F}_{p}$ in form

$$
\mathcal{A}+\mathcal{A}=\mathcal{C} .
$$

Let $g(p)$ denote the cardinality of the largest subset $\mathcal{C}$ of $\mathbb{F}_{p}$ which cannot be represented in this form. Clearly $f(p) \leq g(p)$. Improving on results of Gowers and Green, Alon proved that

$$
p-c_{1} \frac{p^{2 / 3}}{(\log p)^{1 / 3}}<g(p)<p-c_{2} \frac{p^{1 / 2}}{\log p} .
$$

By $f(p) \leq g(p)$ it follows from the upper bound here that

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Note that Green, Gowers and Green, and Alon studied a closely related problem: they studied representations of large subsets $\mathcal{C}$ of $\mathbb{F}_{p}$ in form

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\mathcal{A}+\mathcal{A}=\mathcal{C}
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Let $g(p)$ denote the cardinality of the largest subset $\mathcal{C}$ of $\mathbb{F}_{p}$ which cannot be represented in this form. Clearly $f(p) \leq g(p)$. Improving on results of Gowers and Green, Alon proved that

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p-c_{1} \frac{p^{2 / 3}}{(\log p)^{1 / 3}}<g(p)<p-c_{2} \frac{p^{1 / 2}}{\log p} .
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By $f(p) \leq g(p)$ it follows from the upper bound here that

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& \text { Theorem } 8 \\
& \text { For } p>p_{0} \text { we have } \\
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## 6. On primitive, $k$-primitive, reducible and $k$-reducible subsets of $\mathbb{F}_{p}$

In two papers to be completed soon $K$. Gyarmati and I studied primitive and reducible subsets of $\mathbb{F}_{p}$, the connections between them, and we also introduced and studied further related definitions. First we presented three criteria for primitivity of subsets of $\mathbb{F}_{p}$ (note that while there are several criteria for primitivity of sequences of integers, no criteria have been proved for primitivity of subsets of $\mathbb{F}_{p}$ ).

Theorem 10
Assume that $\mathcal{A}=\left\{a_{1}, a_{2}, \ldots, a_{t}\right\}$ is a subset of $\mathbb{F}_{p}$, and there are $i, j$ with $1 \leq i<j \leq t$ such that $a_{i}+a_{j}-a_{k} \notin \mathcal{A}$ for every $k$ with $1 \leq k \leq t, k \neq i, k \neq j$
and

$$
a_{i}-a_{j}+a_{k} \notin \mathcal{A} \text { for every } k \text { with } 1 \leq k \leq t, k \neq j
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Then $\mathcal{A}$ is primitive.

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To illustrate the applicability of this criterion we showed
If \(p\) is a prime of form \(p=4 k+1\) and \(\mathcal{A} \subset \mathbb{F}_{p}\) is defined by
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\text { Corollary } 2
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$$
\mathcal{A}=\{0,1\} \cup\left\{a \in \mathbb{F}_{p}:\left(\frac{a}{p}\right)=1, \quad\left(\frac{a-1}{p}\right)=-1, \quad a \neq-1, a \neq 2\right\}
$$

then $\mathcal{A}$ is primitive.
It also follows from Theorem 10 that
Corollary 3
If $\mathcal{A} \subset \mathbb{F}_{p}$ is a Sidon set, then it is primitive.

$$
\text { (A set } \mathcal{A}=\left\{a_{1}, a_{2}, \ldots, a_{t}\right\} \text { is called Sidon set if the sums } a_{i}+a_{j} \text { with } 1 \leq i<j \leq t \text { are distinct.) }
$$

The second criterion for primitivity is
Theorem 11
If $\mathcal{A} \subset \mathbb{F}_{p}$ is of the form

$$
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Theorem 11

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#### Abstract




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Corollary 3
If $\mathcal{A} \subset \mathbb{F}_{p}$ is a Sidon set, then it is primitive
(A set $\mathcal{A}=\left\{a_{1}, a_{2}, \ldots, a_{t}\right\}$ is called Sidon set if the sums $a_{i}+a_{j}$ with $1 \leq i<j \leq t$ are distinct.)
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a-a^{\prime}=d, \quad a \in \mathcal{A}, \quad a^{\prime} \in \mathcal{A}
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\max _{d \in \mathbb{F}_{p}^{*}} f(\mathcal{A}, d)<|\mathcal{A}|^{1 / 2}
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Note that Corollary 3 (the primitivity of Sidon sets) also follows from this criterion.
We also proved that Theorem 12 is sharp in the range $0<|\mathcal{A}| \ll p^{1 / 2}$ :
Theorem 13
If $p$ is large enough and $k$ is a positive integer with $k_{0}<k<\frac{1}{2} p^{1 / 4}$, then there is a set $\mathcal{A} \subset \mathbb{F}_{p}$ such that $|\mathcal{A}|=k^{2}$,

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As we have seen, Ostmann's definitions for reducibility and primitivity can be extended to $\mathbb{F}_{p}$ (indeed, these definitions can be used in any additive semigroup). On the other hand, the situation is very much different in case of the definition of totalprimitivity: clearly, this definition cannot be used in its original form in case of finite sets. Instead, we introduced the following definitions:

## Definition 5

If $\mathcal{A}, \mathcal{B} \subset \mathbb{F}_{p}$, then their distance is defined as the cardinality of their symmetric difference and it is denoted by $D(\mathcal{A}, \mathcal{B}):$

$$
D(\mathcal{A}, \mathcal{B})=|(\mathcal{A} \backslash \mathcal{B}) \cup(\mathcal{B} \backslash \mathcal{A})| .
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## Definition 6

For $k \in \mathbb{N}$ a set $\mathcal{A} \subset \mathbb{F}_{p}$ is said to be $k$-primitive if every set $\mathcal{A}^{\prime} \subset \mathbb{F}_{p}$ with $D\left(\mathcal{A}, \mathcal{A}^{\prime}\right) \leq k$ is primitive.
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From Theorem 12 (the third criterion for primitivity) we derived the following criterion for $k$-primitivity:
Theorem 14
Let $\mathcal{A} \subset \mathbb{F}_{p}$ and define $f(\mathcal{A}, d)$ again by $f(\mathcal{A}, d)=\left|\left\{\left(a, a^{\prime}\right): a \in \mathcal{A}, a^{\prime} \in \mathcal{A}, a-a^{\prime}=d\right\}\right|$. If

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\max _{d \in \mathbb{F}_{p}^{*}} f(\mathcal{A}, d)<\frac{1}{2}|\mathcal{A}|^{1 / 2}
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then $\mathcal{A}$ is $k$-primitive.
It follows from this theorem that
Corollary 4
If $\mathcal{A} \subset \mathbb{F}_{p}$ is a Sidon set and $k=\left[\frac{1}{4}|\mathcal{A}|^{1 / 2}\right]$, then $\mathcal{A}$ is $k$-primitive.
If $p$ is a prime then let $M(p)$ denote the greatest integer $k$ such that there is a $k$-primitive set $\mathcal{A}$ in $\mathbb{F}_{p}$. Our next goal was to estimate $M(p)$. We proved:

Theorem 15
For $p \rightarrow \infty$ we have

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(c+o(1)) p<M(p)<\left(\frac{1}{2}+o(1)\right) p
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where $c=0.119 \ldots$ is the smaller zero of the function $\frac{\log 2}{2}+(x \log x+(1-x) \log (1-x))$ in $(0,1)$.

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Next we studied the following problem: if $\mathcal{A}$ is a subset of $\mathbb{F}_{p}$ then, depending on the cardinality of $\mathcal{A}$, what can be said about the size of the greatest reducible subset of $\mathcal{A}$ ?

If $\mathcal{A}$ is a Sidon set, then its subsets are also Sidon sets, thus by Corollary 3 they are primitive so that $\mathcal{A}$ has no reducible subset. Since the cardinality of a Sidon set in $\mathbb{F}_{p}$ can be $\gg p^{1 / 2}$, thus a subset $\mathcal{A} \subset \mathbb{F}_{p}$ of cardinality $\ll p^{1 / 2}$ need not contain a reducible subset. On the other hand, we proved that a subset of cardinality $\gg p^{1 / 2}$ must contain a reducible set. This follows from

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If $\mathcal{A}$ is a subset of $\mathbb{F}_{p}$ with

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\begin{equation*}
\mathcal{B}+\mathcal{C} \text { with }|\mathcal{B}+\mathcal{C}| \geq \min \left\{\frac{|\mathcal{A}|^{2}-|\mathcal{A}|}{p-1}, p\right\},|\mathcal{C}|=2 \tag{4}
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A simple counting argument is used to prove the result.
Observe that the decomposition $\mathcal{B}+\mathcal{C}$ in the last theorem is of very special type: one of two summands is a 2-element subset. One may expect that if $|\mathcal{A}|$ increases, then there are also better balanced decompositions where both $|\mathcal{B}|$ and $|\mathcal{C}|$ are large. Indeed, we have proved such a theorem but before presenting it we need a further definition.

## Definition 7

If $k$ is a positive integer and the set $\mathcal{A} \subset \mathbb{F}_{p}$ has a $k$-decomposition

$$
\left.\mathcal{A}=\mathcal{B}_{1}+\mathcal{B}_{2}+\cdots+\mathcal{B}_{k} \quad \text { (with }\left|\mathcal{B}_{1}\right|,\left|\mathcal{B}_{2}\right|, \ldots,\left|\mathcal{B}_{k}\right| \geq 2\right)
$$

then $\mathcal{A}$ is said to be k-reducible.
We raised two problems on k-reducibility.
(i) Recall that I conjectured that the set of the quadratic residues and the set of the primitive roots are primitive, i.e., they are not 2 -reducible, and as a partial result it has been proved that they are not 3-reducible. Does it not follow from this partial result that they are not 2 -reducible either? We gave a negative answer by constructing subsets of $\mathbb{F}_{p}$ which are 2 -reducible but they are not 3 -reducible.

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#### Abstract

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