On a Hamiltonian Problem For Triple Systems

Andrzej Ruciński
(substituted by M. Schacht)

UAM Poznań and Emory University

joint work with

V. Rödl, M. Schacht and E. Szemerédi

Erdős Centennial Conference

Dirac-type questions

Dirac-type questions

Theorem (Dirac 1952)

If an n-vertex graph G with $n \geq 3$ satisfies $\delta(G) \geq n/2$, then G is Hamiltonian.

Dirac-type questions

Theorem (Dirac 1952)

If an n-vertex graph G with $n \geq 3$ satisfies $\delta(G) \geq n/2$, then G is Hamiltonian.

Main Question

How to extend this result to k-uniform hypergraphs?

Dirac-type questions

Theorem (Dirac 1952)

If an n-vertex graph G with $n \geq 3$ satisfies $\delta(G) \geq n/2$, then G is Hamiltonian.

Main Question

How to extend this result to k-uniform hypergraphs?

Problems:

- What is a (Hamiltonian) cycle in a hypergraph?

Dirac-type questions

Theorem (Dirac 1952)

If an n-vertex graph G with $n \geq 3$ satisfies $\delta(G) \geq n/2$, then G is Hamiltonian.

Main Question

How to extend this result to k-uniform hypergraphs?

Problems:

- What is a (Hamiltonian) cycle in a hypergraph?
- What replaces minimum degree in hypergraphs?

Cycles in hypergraphs

Cycles in hypergraphs

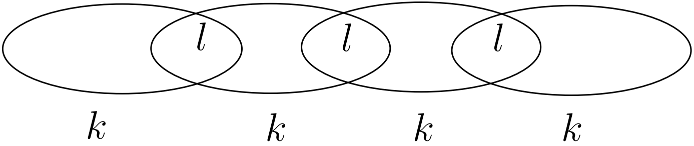
- k-uniform hypergraph $H = (V, E)$, i.e., $E \subseteq \binom{V}{k}$

Cycles in hypergraphs

- k-uniform hypergraph $H = (V, E)$, i.e., $E \subseteq \binom{V}{k}$
- ℓ-overlapping cycle $C_n^{(k,\ell)}$, $0 \le \ell \le k-1$, $(k-\ell)|n$, $|C_n^{(k,\ell)}| = \frac{n}{k-\ell}$

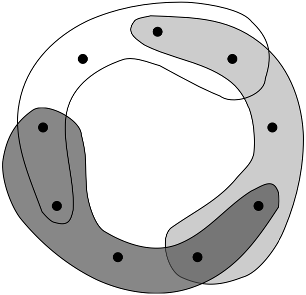
Cycles in hypergraphs

- k-uniform hypergraph $H = (V, E)$, i.e., $E \subseteq \binom{V}{k}$
- ℓ-overlapping cycle $C_n^{(k,\ell)}$, $0 \le \ell \le k-1$, $(k-\ell)|n$, $|C_n^{(k,\ell)}| = \frac{n}{k-\ell}$



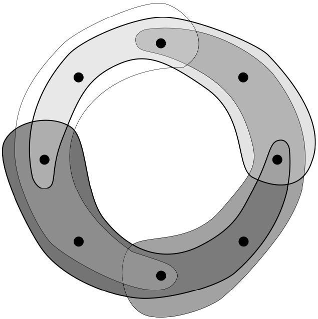
Cycles in hypergraphs

- k-uniform hypergraph $H = (V, E)$, i.e., $E \subseteq \binom{V}{k}$
- ℓ-overlapping cycle $C_n^{(k,\ell)}$, $0 \le \ell \le k-1$, $(k-\ell)|n$, $|C_n^{(k,\ell)}| = \frac{n}{k-\ell}$



Cycles in hypergraphs

- k-uniform hypergraph $H = (V, E)$, i.e., $E \subseteq \binom{V}{k}$
- ℓ-overlapping cycle $C_n^{(k,\ell)}$, $0 \le \ell \le k-1$, $(k-\ell)|n$, $|C_n^{(k,\ell)}| = \frac{n}{k-\ell}$



Degrees in hypergraphs

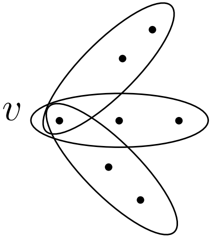
- minimum vertex degree $\delta_1(H)$

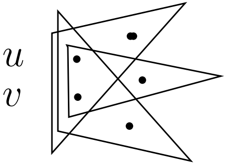
Degrees in hypergraphs

- minimum vertex degree $\delta_1(H)$
- minimum pair degree $\delta_2(H)$

Degrees in hypergraphs

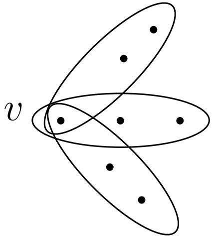
- minimum vertex degree $\delta_1(H)$
- minimum pair degree $\delta_2(H)$

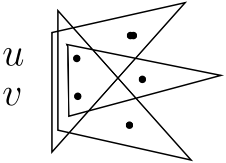




Degrees in hypergraphs

- minimum vertex degree $\delta_1(H)$
- minimum pair degree $\delta_2(H)$





- triple degrees $\delta_3(H)$
- ...

Main question

Theorem (Dirac 1952)

If an n-vertex graph G satisfies $\delta(G) \geq n/2$, then G is Hamiltonian.

Main question

Theorem (Dirac 1952)

If an n-vertex graph G satisfies $\delta(G) \geq n/2$, then G is Hamiltonian.

Main question

Given integers k, ℓ, and d determine the function $h_d^{(k,\ell)}(n)$ with the property

$$\delta_d(H) \geq h_d^{(k,\ell)}(n) \quad \Longrightarrow \quad \text{Hamiltonian } \ell\text{-cycle in } H$$

for any n-vertex k-uniform hypergraph H.

Main question

Theorem (Dirac 1952)

If an n-vertex graph G satisfies $\delta(G) \geq n/2$, then G is Hamiltonian.

Main question

Given integers k, ℓ, and d determine the function $h_d^{(k,\ell)}(n)$ with the property

$$\delta_d(H) \geq h_d^{(k,\ell)}(n) \quad \Longrightarrow \quad \text{Hamiltonian } \ell\text{-cycle in } H$$

for any n-vertex k-uniform hypergraph H.

Remarks:

- $\ell = 0 \rightarrow$ perfect matchings

Main question

Theorem (Dirac 1952)

If an n-vertex graph G satisfies $\delta(G) \geq n/2$, then G is Hamiltonian.

Main question

Given integers k, ℓ, and d determine the function $h_d^{(k,\ell)}(n)$ with the property

$$\delta_d(H) \geq h_d^{(k,\ell)}(n) \quad \Longrightarrow \quad \text{Hamiltonian } \ell\text{-cycle in } H$$

for any n-vertex k-uniform hypergraph H.

Remarks:

- $\ell = 0 \to$ perfect matchings
- Bollobás, Daykin & Erdős 1976

Main question

Theorem (Dirac 1952)

If an n-vertex graph G satisfies $\delta(G) \geq n/2$, then G is Hamiltonian.

Main question

Given integers k, ℓ, and d determine the function $h_d^{(k,\ell)}(n)$ with the property

$$\delta_d(H) \geq h_d^{(k,\ell)}(n) \quad \Longrightarrow \quad \text{Hamiltonian } \ell\text{-cycle in } H$$

for any n-vertex k-uniform hypergraph H.

Remarks:

- $\ell = 0 \to$ perfect matchings
- Bollobás, Daykin & Erdős 1976
  Daykin & Häggkvist 1981

$h_1^{(3,0)}(n)$

Main question

Theorem (Dirac 1952)

If an n-vertex graph G satisfies $\delta(G) \geq n/2$, then G is Hamiltonian.

Main question

Given integers k, ℓ, and d determine the function $h_d^{(k,\ell)}(n)$ with the property

$$\delta_d(H) \geq h_d^{(k,\ell)}(n) \quad \Longrightarrow \quad \text{Hamiltonian } \ell\text{-cycle in } H$$

for any n-vertex k-uniform hypergraph H.

Remarks:

- $\ell = 0 \rightarrow$ perfect matchings
- Bollobás, Daykin & Erdős 1976
  Daykin & Häggkvist 1981 — $h_1^{(3,0)}(n)$
- G. Y. Katona and Kierstead 1999 — $h_2^{(3,2)}(n)$

Main question

Theorem (Dirac 1952)

If an n-vertex graph G satisfies $\delta(G) \geq n/2$, then G is Hamiltonian.

Main question

Given integers k, ℓ, and d determine the function $h_d^{(k,\ell)}(n)$ with the property

$$\delta_d(H) \geq h_d^{(k,\ell)}(n) \quad \Longrightarrow \quad \text{Hamiltonian } \ell\text{-cycle in } H$$

for any n-vertex k-uniform hypergraph H.

Remarks:

- $\ell = 0 \rightarrow$ perfect matchings
- Bollobás, Daykin & Erdős 1976
  Daykin & Häggkvist 1981 — $h_1^{(3,0)}(n)$
- G. Y. Katona and Kierstead 1999 — $h_2^{(3,2)}(n)$
- Today: $k = 3$ and $\ell = 1$ or 2

Main question

Theorem (Dirac 1952)

If an n-vertex graph G satisfies $\delta(G) \geq n/2$, then G is Hamiltonian.

Main question

Given integers k, ℓ, and d determine the function $h_d^{(k,\ell)}(n)$ with the property

$$\delta_d(H) \geq h_d^{(k,\ell)}(n) \quad \Longrightarrow \quad \text{Hamiltonian } \ell\text{-cycle in } H$$

for any n-vertex k-uniform hypergraph H.

Remarks:

- $\ell = 0 \rightarrow$ perfect matchings
- Bollobás, Daykin & Erdős 1976
  Daykin & Häggkvist 1981 — $h_1^{(3,0)}(n)$
- G. Y. Katona and Kierstead 1999 — $h_2^{(3,2)}(n)$
- Today: $k = 3$ and $\ell = 1$ or 2 and we write $h_d^\ell = h_d^{(3,\ell)}$

Some known results

Theorems ($d = 2$, n large)

Some known results

Theorems ($d = 2$, n large)

- $h_2^1(n) \sim n/4$ Kühn and Osthus

Some known results

Theorems ($d = 2$, n large)

- $h_2^1(n) \sim n/4$ — Kühn and Osthus
- $h_2^2(n) = \lfloor n/2 \rfloor$ — Rödl, R. & Szemerédi

Some known results

Theorems ($d = 2$, n large)

- $h_2^1(n) \sim n/4$ — Kühn and Osthus
- $h_2^2(n) = \lfloor n/2 \rfloor$ — Rödl, R. & Szemerédi

Theorems ($d = 1$, n large)

Some known results

Theorems ($d=2$, n large)

- $h_2^1(n) \sim n/4$ — Kühn and Osthus
- $h_2^2(n) = \lfloor n/2 \rfloor$ — Rödl, R. & Szemerédi

Theorems ($d=1$, n large)

- $h_1^1(n) \sim \frac{7}{16}\binom{n}{2}$ — Buß, Hàn & Schacht

Some known results

Theorems ($d = 2$, n large)

- $h_2^1(n) \sim n/4$ — Kühn and Osthus
- $h_2^2(n) = \lfloor n/2 \rfloor$ — Rödl, R. & Szemerédi

Theorems ($d = 1$, n large)

- $h_1^1(n) \sim \frac{7}{16}\binom{n}{2}$ — Buß, Hàn & Schacht
- $h_1^2(n) \le (1 - 5 \cdot 10^{-7})\binom{n}{2}$ — Glebov, Person & Weps

Some known results

Theorems ($d = 2$, n large)

- $h_2^1(n) \sim n/4$ — Kühn and Osthus
- $h_2^2(n) = \lfloor n/2 \rfloor$ — Rödl, R. & Szemerédi

Theorems ($d = 1$, n large)

- $h_1^1(n) \sim \frac{7}{16}\binom{n}{2}$ — Buß, Hàn & Schacht
- $h_1^2(n) \leq (1 - 5 \cdot 10^{-7})\binom{n}{2}$ — Glebov, Person & Weps
- $h_1^2(n) \leq \left(\frac{5-\sqrt{5}}{3}\right)\binom{n}{2} \approx 0.92\binom{n}{2}$ — Rödl & R.

Some known results

Theorems ($d = 2$, n large)

- $h_2^1(n) \sim n/4$ — Kühn and Osthus
- $h_2^2(n) = \lfloor n/2 \rfloor$ — Rödl, R. & Szemerédi

Theorems ($d = 1$, n large)

- $h_1^1(n) \sim \frac{7}{16}\binom{n}{2}$ — Buß, Hàn & Schacht
- $h_1^2(n) \leq (1 - 5 \cdot 10^{-7})\binom{n}{2}$ — Glebov, Person & Weps
- $h_1^2(n) \leq \left(\frac{5-\sqrt{5}}{3}\right)\binom{n}{2} \approx 0.92\binom{n}{2}$ — Rödl & R.

New bound (work in progress)

$$h_1^2(n) \lesssim \frac{4}{5}\binom{n}{2}$$

Some known results

Theorems ($d = 2$, n large)

- $h^1_2(n) \sim n/4$ — Kühn and Osthus
- $h^2_2(n) = \lfloor n/2 \rfloor$ — Rödl, R. & Szemerédi

Theorems ($d = 1$, n large)

- $h^1_1(n) \sim \frac{7}{16}\binom{n}{2}$ — Buß, Hàn & Schacht
- $h^2_1(n) \leq (1 - 5 \cdot 10^{-7})\binom{n}{2}$ — Glebov, Person & Weps
- $h^2_1(n) \leq \left(\frac{5-\sqrt{5}}{3}\right)\binom{n}{2} \approx 0.92\binom{n}{2}$ — Rödl & R.

New bound (work in progress)

$$h^2_1(n) \lesssim \frac{4}{5}\binom{n}{2}$$

Where do we stand?

Where do we stand?

New bound (work in progress)

$$h_1^2(n) \lesssim \frac{4}{5}\binom{n}{2}$$

Where do we stand?

New bound (work in progress)

$$h_1^2(n) \lesssim \frac{4}{5}\binom{n}{2}$$

Conjecture

$$h_1^2(n) \sim h_1^0(n) \sim \frac{5}{9}\binom{n}{2}$$

Where do we stand?

New bound (work in progress)

$$h_1^2(n) \lesssim \frac{4}{5}\binom{n}{2}$$

Conjecture

$$h_1^2(n) \sim h_1^0(n) \sim \frac{5}{9}\binom{n}{2}$$

Some evidence:

Where do we stand?

New bound (work in progress)

$$h_1^2(n) \lesssim \frac{4}{5}\binom{n}{2}$$

Conjecture

$$h_1^2(n) \sim h_1^0(n) \sim \frac{5}{9}\binom{n}{2}$$

Some evidence:

- $h_2^2(n) \sim h_2^0(n) \sim n/2$

Where do we stand?

New bound (work in progress)

$$h_1^2(n) \lesssim \frac{4}{5}\binom{n}{2}$$

Conjecture

$$h_1^2(n) \sim h_1^0(n) \sim \frac{5}{9}\binom{n}{2}$$

Some evidence:

- $h_2^2(n) \sim h_2^0(n) \sim n/2$
- $h_2^1(n) \sim n/4 \sim$"min. pair degree for matchings of size $n/4$"

Where do we stand?

New bound (work in progress)

$$h_1^2(n) \lesssim \frac{4}{5}\binom{n}{2}$$

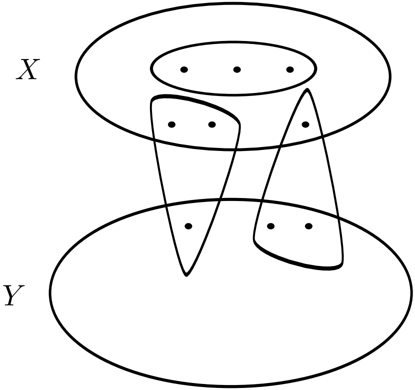
Conjecture

$$h_1^2(n) \sim h_1^0(n) \sim \frac{5}{9}\binom{n}{2}$$

Some evidence:

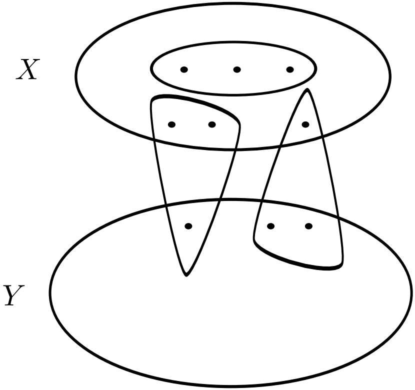
- $h_2^2(n) \sim h_2^0(n) \sim n/2$
- $h_2^1(n) \sim n/4 \sim$"min. pair degree for matchings of size $n/4$"
- $h_1^1(n) \sim \frac{7}{16}\binom{n}{2} \sim$"min. vertex degree for matchings of size $n/4$"

Lower bound construction



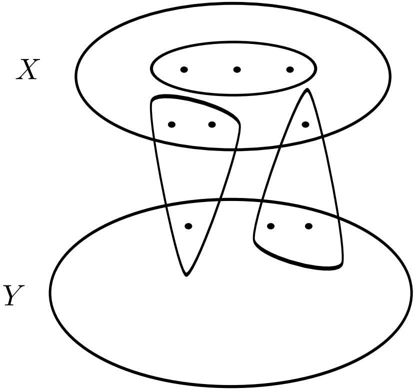
- Suppose $3|n$ and $|X| = n/3 - 1$

Lower bound construction



- Suppose $3|n$ and $|X| = n/3 - 1$
- $e \in E(H) \iff e \cap X \neq \emptyset$

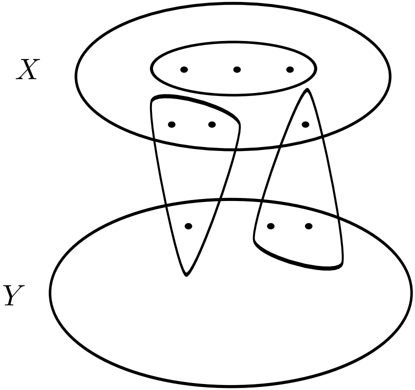
Lower bound construction



- Suppose $3|n$ and $|X| = n/3 - 1$
- $e \in E(H) \iff e \cap X \neq \emptyset$

$\Rightarrow \delta_1(H) \sim \binom{n}{2} - \binom{2n/3}{2} \sim \frac{5}{9}\binom{n}{2}$,

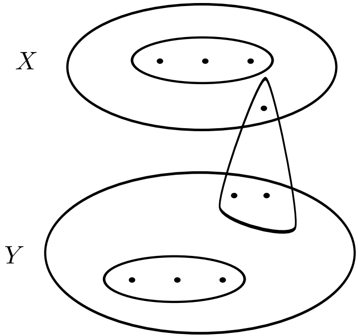
Lower bound construction



- Suppose $3|n$ and $|X| = n/3 - 1$
- $e \in E(H) \iff e \cap X \neq \emptyset$

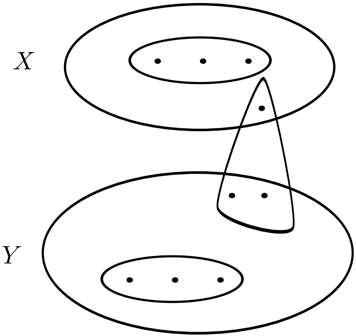
$\Rightarrow$ $\delta_1(H) \sim \binom{n}{2} - \binom{2n/3}{2} \sim \frac{5}{9}\binom{n}{2}$, but H contains no perfect matching

Lower bound construction with perfect matching



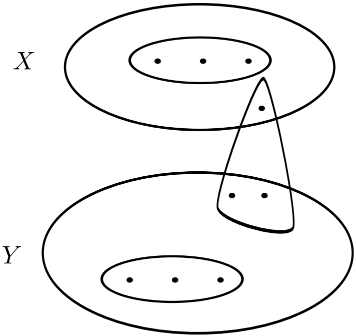
- Suppose $3|n$ and $|X| = n/3 + 1$

Lower bound construction with perfect matching



- Suppose $3|n$ and $|X| = n/3 + 1$
- $e \in E(H) \iff |e \cap Y| \neq 1$

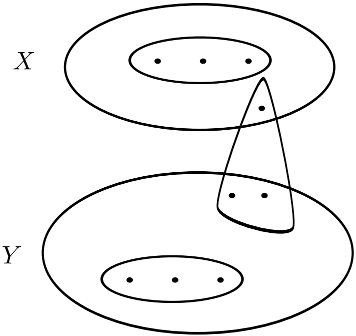
Lower bound construction with perfect matching



- Suppose $3|n$ and $|X| = n/3 + 1$
- $e \in E(H) \iff |e \cap Y| \neq 1$

$\Rightarrow$ $\delta_1(H) \sim \binom{n}{2} - \frac{n}{3} \cdot \frac{2n}{3} \sim \frac{5}{9}\binom{n}{2}$,

Lower bound construction with perfect matching

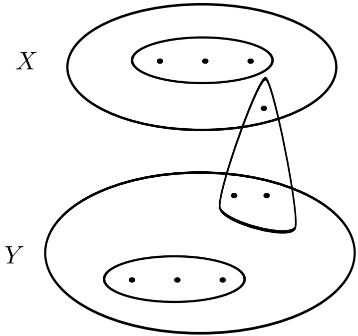


- Suppose $3|n$ and $|X| = n/3 + 1$
- $e \in E(H) \quad \Longleftrightarrow \quad |e \cap Y| \neq 1$

$\Rightarrow$ $\delta_1(H) \sim \binom{n}{2} - \frac{n}{3} \cdot \frac{2n}{3} \sim \frac{5}{9}\binom{n}{2}$,

- every edge with two vertices in $|X|$ is contained in X

Lower bound construction with perfect matching



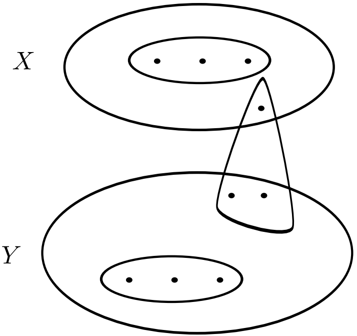
- Suppose $3|n$ and $|X| = n/3 + 1$
- $e \in E(H) \iff |e \cap Y| \neq 1$

$\Rightarrow$ $\delta_1(H) \sim \binom{n}{2} - \frac{n}{3} \cdot \frac{2n}{3} \sim \frac{5}{9}\binom{n}{2}$,

- every edge with two vertices in $|X|$ is contained in X

$\Rightarrow$ every edge of a C_n^2 intersects Y in at least two vertices

Lower bound construction with perfect matching



- Suppose $3|n$ and $|X| = n/3 + 1$
- $e \in E(H) \iff |e \cap Y| \neq 1$

$\Rightarrow$ $\delta_1(H) \sim \binom{n}{2} - \frac{n}{3} \cdot \frac{2n}{3} \sim \frac{5}{9}\binom{n}{2}$,

- every edge with two vertices in $|X|$ is contained in X

$\Rightarrow$ every edge of a C_n^2 intersects Y in at least two vertices $\lightning$

Absorbing method

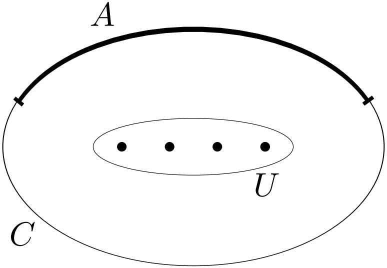
1. Find an absorbing path A in H with $|V(A)| = c_1 n$:

Absorbing method

1. Find an absorbing path A in H with $|V(A)| = c_1 n$:
   $\forall U \subseteq V \setminus V(A)$ with $|U| \leq c_2 n$ $(\ll c_1 n)$
   $\exists$ path A_U with same endpairs and $V(A_U) = V(A) \cup U$.

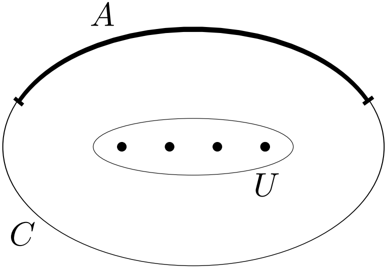
Absorbing method

1. Find an absorbing path A in H with $|V(A)| = c_1 n$:
   $\forall U \subseteq V \setminus V(A)$ with $|U| \leq c_2 n$ ($\ll c_1 n$)
   $\exists$ path A_U with same endpairs and $V(A_U) = V(A) \cup U$.
2. Find almost Hamiltonian cycle C containing A.



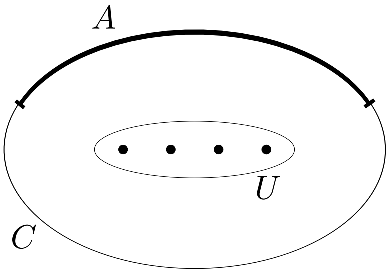
Absorbing method

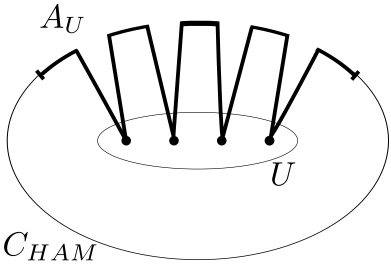
1. Find an absorbing path A in H with $|V(A)| = c_1 n$:
   $\forall U \subseteq V \setminus V(A)$ with $|U| \leq c_2 n$ ($\ll c_1 n$)
   $\exists$ path A_U with same endpairs and $V(A_U) = V(A) \cup U$.
2. Find almost Hamiltonian cycle C containing A.
3. Apply absorbing property of A to $U = V \setminus V(C)$ and obtain Hamiltonian cycle.



Absorbing method

1. Find an absorbing path A in H with $|V(A)| = c_1 n$:
   $\forall U \subseteq V \setminus V(A)$ with $|U| \leq c_2 n$ ($\ll c_1 n$)
   $\exists$ path A_U with same endpairs and $V(A_U) = V(A) \cup U$.
2. Find almost Hamiltonian cycle C containing A.
3. Apply absorbing property of A to $U = V \setminus V(C)$ and obtain Hamiltonian cycle.

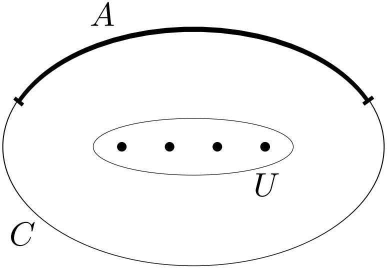


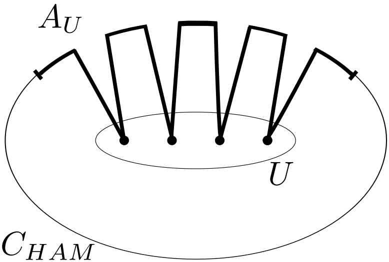


HAA. . . AM

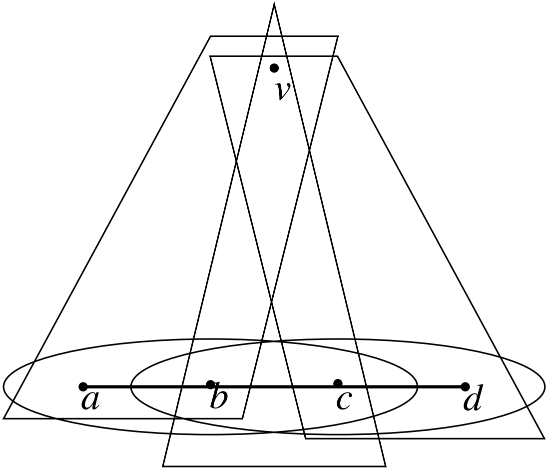
Absorbing method

1. Find an absorbing path A in H with $|V(A)| = c_1 n$:
   $\forall U \subseteq V \setminus V(A)$ with $|U| \leq c_2 n$ ($\ll c_1 n$)
   $\exists$ path A_U with same endpairs and $V(A_U) = V(A) \cup U$.
2. Find almost Hamiltonian cycle C containing A.
3. Apply absorbing property of A to $U = V \setminus V(C)$ and obtain Hamiltonian cycle.

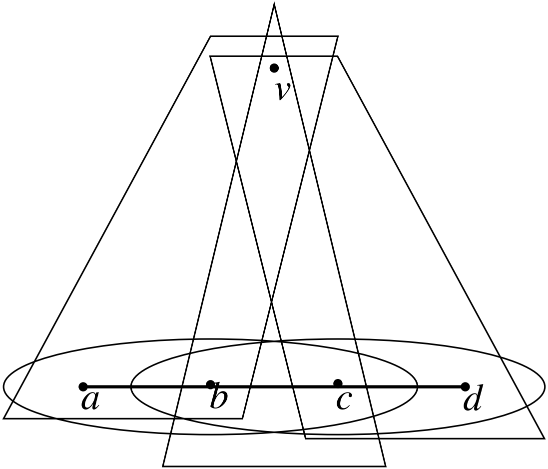




Finding absorbers with $\delta_2(H) \geq (1/2 + \varepsilon)n$



Finding absorbers with $\delta_2(H) \geq (1/2 + \varepsilon)n$



Fact: For every $v \in V$ there are $\varepsilon^2 n^4$ absorbers (a, b, c, d).

Absorbing path

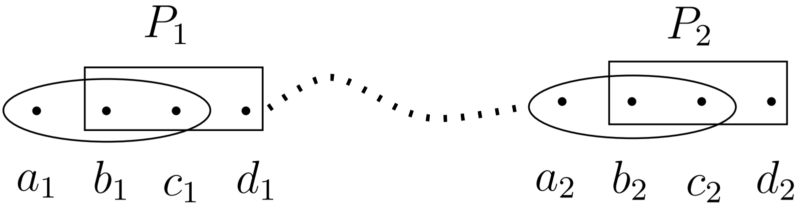
1. Randomly select γn from all 4-tuples

Absorbing path

1. Randomly select γn from all 4-tuples
2. Select a pairwise disjoint subset of those 4-tuples forming paths
   $\Rightarrow$ w.h.p. for every $v \in V$ at least $\gamma^4 \varepsilon^2 n$ absorber were selected

Absorbing path

1. Randomly select γn from all 4-tuples
2. Select a pairwise disjoint subset of those 4-tuples forming paths
   $\Rightarrow$ w.h.p. for every $v \in V$ at least $\gamma^4 \varepsilon^2 n$ absorber were selected
3. Connect the selected 4-tuples P_i to obtain the path A



Some ideas

- remove hyperedges from H, that contain pairs (x, y) with $\deg_H(x, y) \leq (1/2 + \varepsilon)n$

Some ideas

- remove hyperedges from H, that contain pairs (x, y) with $\deg_H(x, y) \leq (1/2 + \varepsilon)n$
  $\leftarrow$ requires $\delta_1(H) \geq \frac{5-\sqrt{5}}{3}\binom{n}{2}$

Some ideas

- remove hyperedges from H, that contain pairs (x, y) with $\deg_H(x, y) \leq (1/2 + \varepsilon)n$
  $\leftarrow$ requires $\delta_1(H) \geq \frac{5-\sqrt{5}}{3}\binom{n}{2}$

- slightly more careful, remove only hyperedges which contain no pair of high degree

Some ideas

- remove hyperedges from H, that contain pairs (x, y) with $\deg_H(x, y) \leq (1/2 + \varepsilon)n$
  $\leftarrow$ requires $\delta_1(H) \geq \frac{5-\sqrt{5}}{3}\binom{n}{2}$
- slightly more careful, remove only hyperedges which contain no pair of high degree
- balance between "finding absorbers" and "making connections between large pairs"

Questions