The phase transition in Achlioptas processes

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Outline

The Erdős–Rényi random graph process

Fix a set V of n vertices (usually $V = [n] = \{1, 2, \ldots, n\}$).
Define a random sequence (G_m), $m = 0, 1, .., \binom{n}{2}$, of graphs on V as follows:

- G_0 has no edges.
- Given $G_0, \ldots, G_{m-1}$, pick e_m uniformly at random from all edges not present in G_{m-1}.
- Set $G_m = G_{m-1} + e_m$.

Note G_m has m edges and the distribution of $G_{n,m}$.

Erdős–Rényi phase transition

Classical result of Erdős and Rényi (1960).
Let $L_1(G)$ denote the number of vertices in the largest component of G.

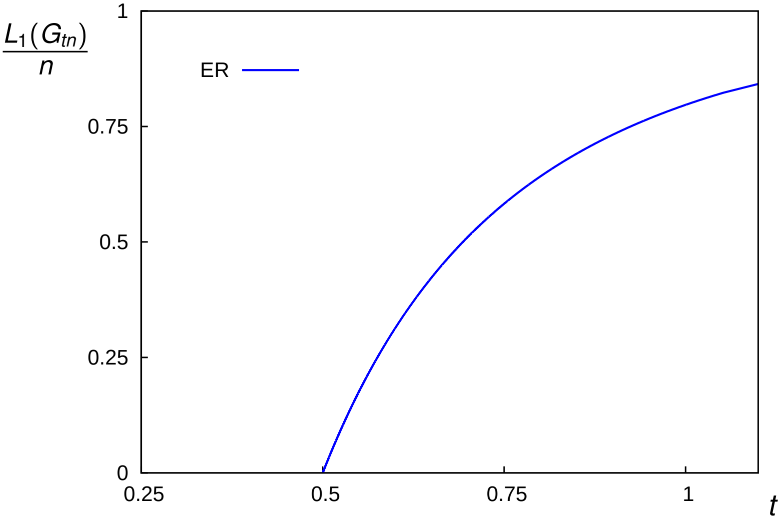
Theorem

Let t be a constant, and let $G = G_{n,tn}$. (Ignore rounding.)

- *If $t < 1/2$ then $\exists$ constant $A(t) > 0$ such that* $$L_1(G) \leqslant A(t) \log n \text{ whp (with high probability).}$$
- *If $t > 1/2$ then $\exists$ constant $\rho(t)$ such that $L_1(G)/n \xrightarrow{\text{p}} \rho(t)$.*
- *If $t = 1/2$ then $L_1(G)$ is of order $n^{2/3}$ whp.*

They gave formulae for the constants, and proved more.

Erdős–Rényi scaling limit



Where next?

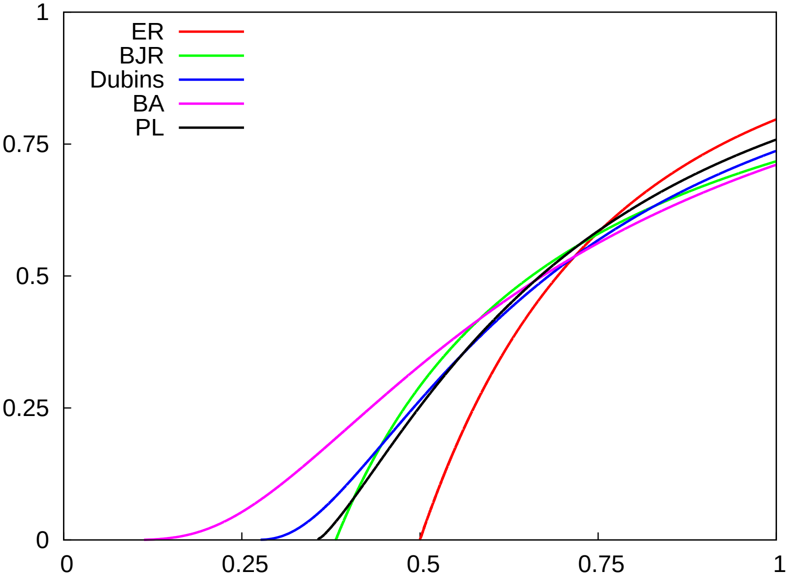
Two directions from here:

- Investigate this model in more detail.
- Investigate this phenomenon in other models.

Phase transitions in other models

- Random subgraphs of hypercubes. Ajtai, Komlós, Szemerédi, 1982
- Random subgraphs of r-regular graphs. Goerdt, 2001
- Bollobás's configuration model: Molloy, Reed, 1995,1998
  scale-free case: Chung, Lu, 2000/1
- Barabási–Albert model. Bollobás, R., 2003
- Uniformly grown (CHKNS/Dubins) model. Durrett, 2003, Bollobás, Janson, R., 2005
- BJR model Bollobás, Janson, R., 2007
- (Classical percolation: Hammersley, Harris, Kesten, ...)

Various BJR scaling limits



Power of two random choices

Suppose we have n (or $\Theta(n)$) balls to place into n bins. (Corresponding e.g., to hashing, or task allocation.)

Random choice: fairly uniform, but still $\log n / \log\log n$ balls in some bin.

For each ball, choose two random bins, put ball in least full. Then much more uniform: at most $\log\log n$ balls in any bin.
Azar, Broder, Karlin, Upfal, 1994, 1999

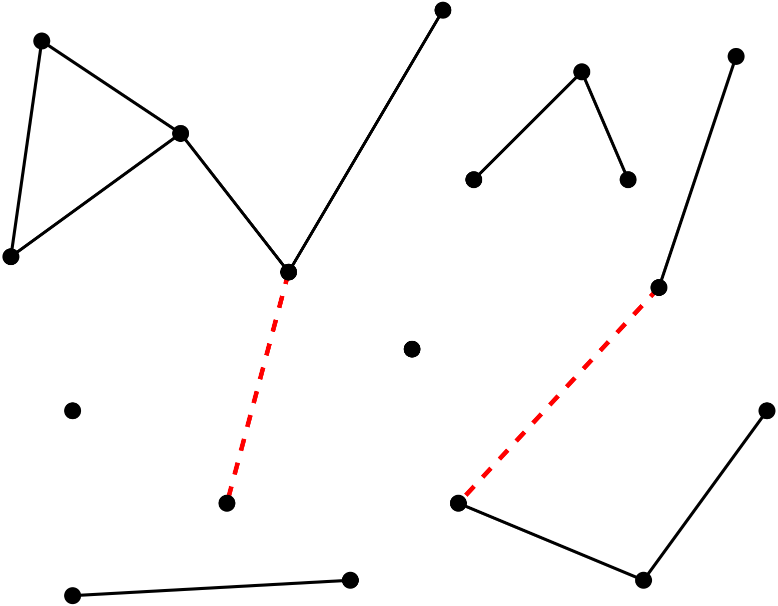
In 2000 Achlioptas asked: can two choices affect the ER random graph process?

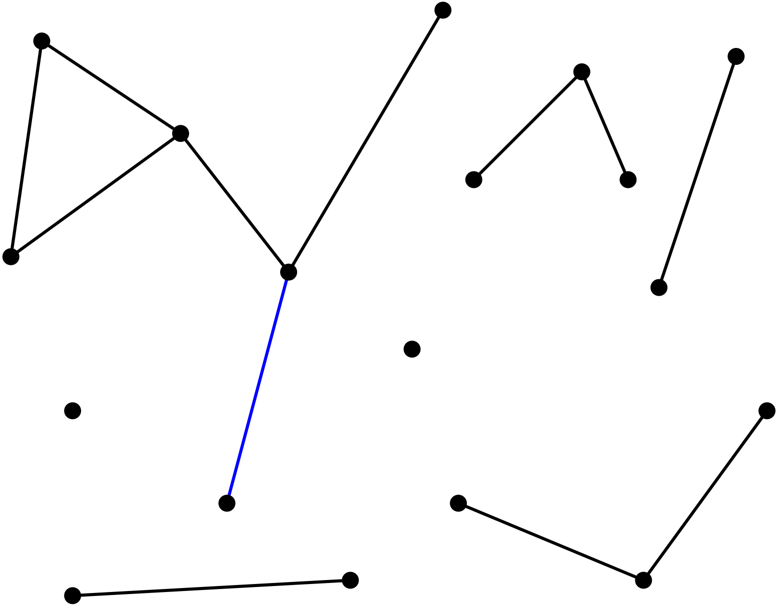
Can it delay/accelerate the phase transition?

Semi-formal definition

Let $\mathcal{R}$ be a 'decision rule'.
Let V be a set of n vertices. The Achlioptas process $(G_m)_{m\geqslant 0} = (G^{\mathcal{R}}_{n,m})_{m\geqslant 0}$ is defined as follows:

- G_0 is the graph on V with no edges.
- Given $G_0, \ldots, G_{m-1}$, let e^1_m and e^2_m be two possible edges chosen uniformly at random (independently).
- Select one of the edges according to rule $\mathcal{R}$.
- Then $G_m = G_{m-1} + e^1_m$ or $G_{m-1} + e^2_m$.





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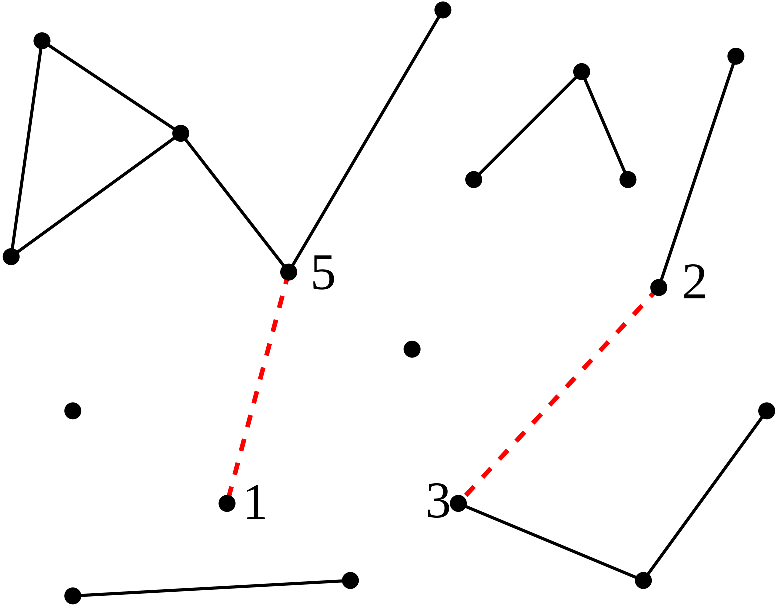
- Then $G_m = G_{m-1} + e^1_m$ or $G_{m-1} + e^2_m$.

Minor variants: select from edges not present/not previously selected. Or select random vertices. Here, no significant difference.

Classes of rules

Classes of rules (more to less general):

- General: decision based on any information about past+present (i.e., $(e^i_k)_{k \leqslant m, i=1,2}$ - in particular on G_{m-1} as labelled graph).
- Size rules: decision based only on sizes of components involved.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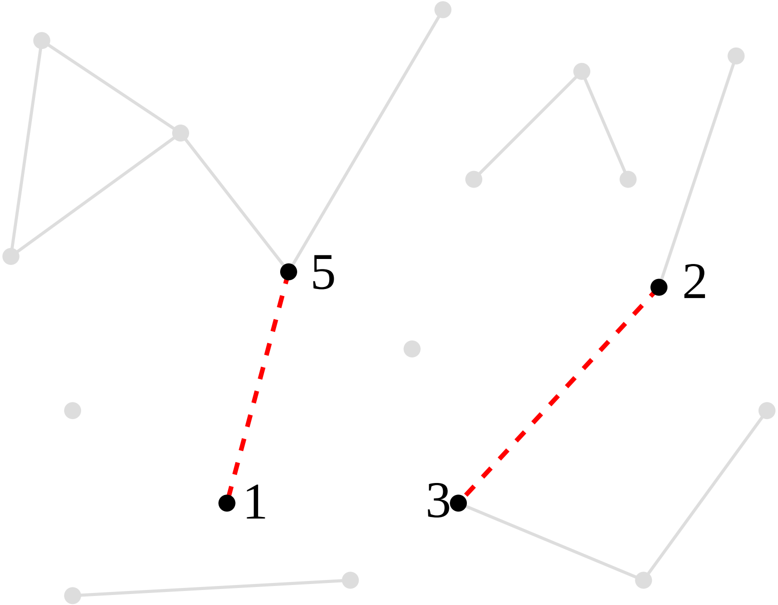
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- Size rules: decision based only on sizes of components involved.
- Bounded size rules: as above, but for constant B all sizes $> B$ treated same.

Most of the time only size rules considered. Bounded size easier to analyze.

Which rule?

How to accelerate/delay phase transition?

A key statistic is the susceptibility:

$$\chi(G) = \frac{1}{n} \sum_i |C_i|^2.$$

If we join components of sizes a and b, $\Delta\chi(G) \propto ab$.

Bollobás suggested product rule should be best.

Earlier phase transition

With hindsight, trivial to produce transition before $t = 0.5$.

E.g., let $U \subset V$ with $|U| = 0.99n$, and select edges within U when possible.

Edges are added to U at rate > 0.999, so Erdős–Rényi result gives giant by $m = (0.5n * 0.99)/0.999 < 0.5n$.

In general ‘inhomogeneous’ (but ‘blind’) rules give earlier transition.

Isolated vertex rules

Bohman and Frieze (2001) found a rule $\mathcal{R}$ for which they could prove $t_c > 0.5$. (In fact $t_c > 0.535$).

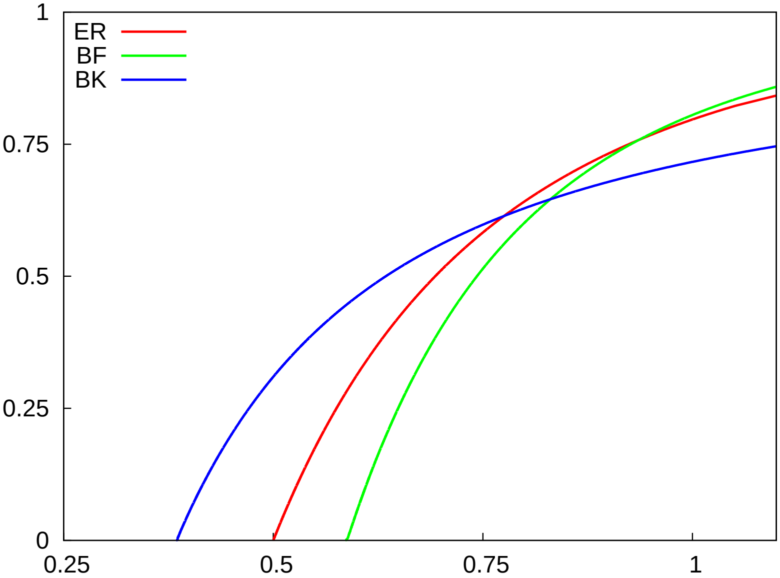
Essentially: 'select e_m^1 if both ends isolated, otherwise e_m^2'.

A bounded size rule, with $B = 1$.

Bohman and Kravitz (2006) proved $t_c < 0.385$ for the size rule 'select e_m^1 is nether end isolated, otherwise e_m^2'.

They also analyzed other 'bounded first-edge size rules'.

Some scaling limits



Bounded size rules

Each size rule $\mathcal{R}$ suggests a system of differential equations:

$$\rho_k'(t) = \sum_{a,b,c,d} \Delta_{k,a,b,c,d}\ \rho_a(t)\rho_b(t)\rho_c(t)\rho_d(t)$$

with $\Delta_{k,a,b,c,d} \in \{-2k, -k, 0, k\}$ depending on $\mathcal{R}$.

Spencer and Wormald 2007: for bounded size rules, the small components behave as the DEs predict.
Also, t_c (point when giant appears) given by the blow-up point of the system of DEs.

Janson and Spencer 2012: found rate of emergence of giant for Bohman–Frieze process.

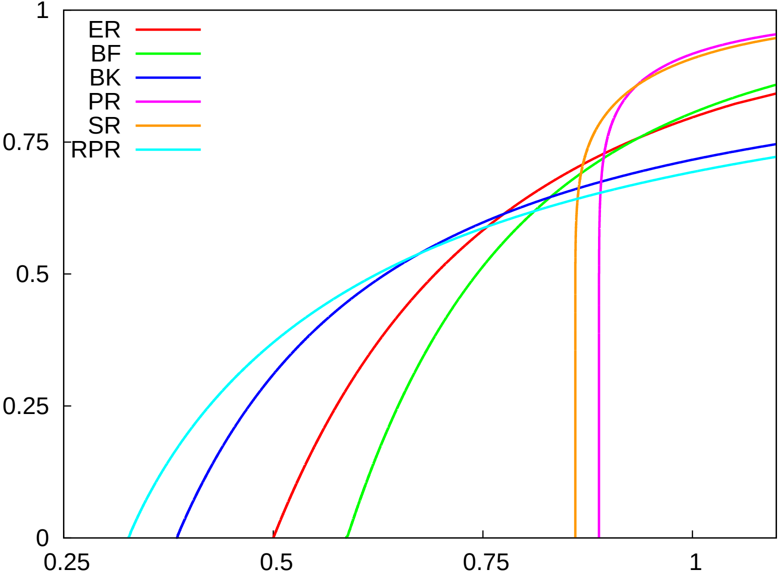
Questions

Even for bounded size rules, does scaling limit exist?

Are all vertices not in small components in a unique giant?

What about non-bounded case, e.g., Bollobás's product rule?

More scaling limits



Explosive percolation

Conjecture (Achlioptas, D'Souza, Spencer 2009)

The product rule has a discontinuous transition!
More precisely, $L_1(G_m)$ *grows from* $< n^{1/2}$ *to* $> 0.5n$ *in* $< 2n^{2/3}$ *steps.*

Based on 'conclusive' numerical evidence.

Called explosive percolation.

Very exciting - many citations in physics literature.

Heuristic support from many physicists. But not all: da Costa, Dorogovtsev, Goltsev, Mendes 2010

Theorem (R., Warnke 2011)

Every Achlioptas process has a continuous phase transition.

Theorem (R., Warnke 2011)

In any Achlioptas process, almost all vertices not in small components are in a unique giant.

Analogue of Erdős–Rényi sprinkling argument.

Corollary

If small components 'behave', scaling limit exists. In particular, all bounded size rules have a scaling limit.

Convergence questions

Does the product rule have a scaling limit? Perhaps all size rules do?

Theorem (R., Warnke 2013+)

Small components behave at least until the point where $\chi(G_m)$ diverges.

Probably, this point is t_c.

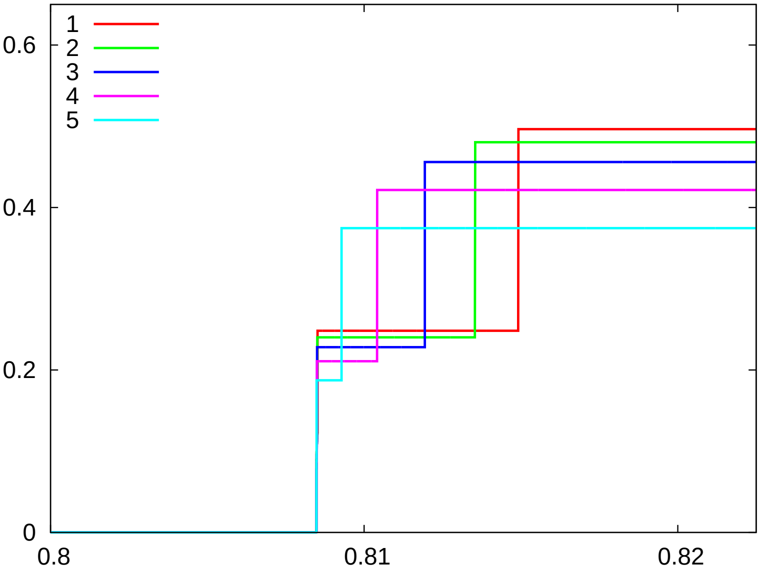
Theorem (R., Warnke 2013+)

Yes (modulo natural condition) if a corresponding system of differential equations has a unique solution.

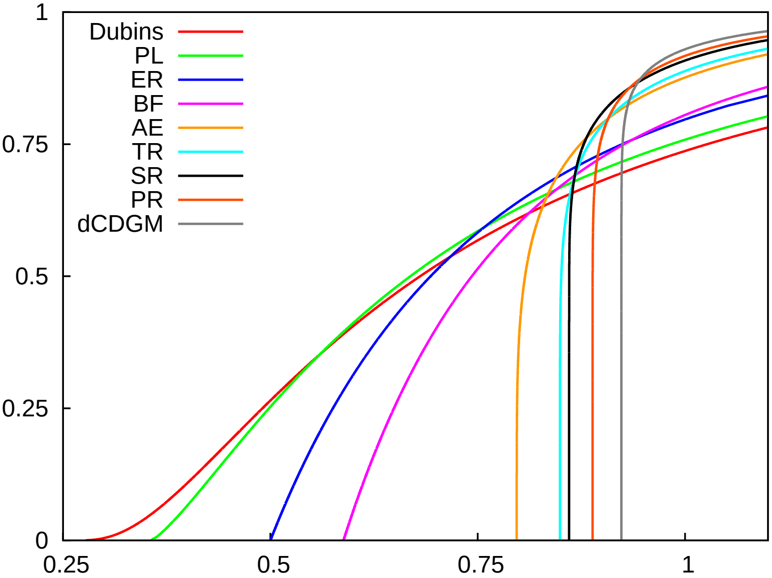
NB. System of DEs includes term for giant component.

'Direct' proof - not via standard DE method.

A warning!



What is happening?



What is happening?

For a class of unbounded size rules, the phase transition appears to be extremely steep.

Can we show $\rho'(t_c) = \infty$?

Can we give sensible upper bounds? R-W result gives $\rho(t_c + \varepsilon) \leqslant 1/(\log\log(1/\varepsilon))^C$.

Physicists expect $\rho(t_c + \varepsilon) = \Theta(\varepsilon^\beta)$. Perhaps $\beta = 1/18$ for dCDGM rule.

Other questions

Can we examine the scaling window, say for bounded size rules?

- Bhamidi, Budhiraja, Wang
- Kang, Perkins, Spencer

What about other explosive processes?

- Panagiotou, Spöhel, Steger, Thomas 2013
- Cho, Hwang, Herrmann, Kahng 2013