PAUL ERDŐS AND THE DIFFERENCE OF PRIMES

by JÁNOS PINTZ

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$$\mu(n) = \begin{cases} (-1)^m & \text{if } n \text{ is square-free and has } m \text{ prime factors} \\ 0 & \text{otherwise} \end{cases}$$

1. LARGE GAPS BETWEEN CONSECUTIVE PRIMES

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Backlund	(1929):	$\lambda \ge 2$
Brauer–Zeitz	(1930):	$\lambda \ge 4$

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Rankin (1938): $\limsup_{n\to\infty} \frac{d_n (\log_3 n)^2}{\log n \log_2 n \log_4 n} \ge C_1 = \frac{1}{3}$

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Maier–Pomerance (1990): $C_1 = 1.31\ldots e^\gamma$

J. Pintz (1997): $C_1 = 2e^\gamma$

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REMARK: Bounded gaps conjecture $\Leftrightarrow$ There is at least one Polignac number $\Leftrightarrow \exists k \in \mathbb{Z}^+$: $d_n = 2k$ i.o.

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H. Maier (1988): $\Delta < 0.2486 \cdots < 1/4$

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J. Pintz (2011–2013): $d_n < C(\log n)^{3/7}(\log\log n)^{4/7}$ i.o.
and this is *the limit of the original GPY-method* (without some sort of improvement of the Bombieri–Vinogradov theorem) as shown by **B. Farkas – J. Pintz – Sz. Gy. Révész (2013)**

Definition: Primes have an *admissible distribution level* ϑ

$$(1)\qquad \sum_{q \le X^{\vartheta-\varepsilon}} \max_{\substack{a \\ (a,q)=1}} \Bigg| \sum_{\substack{p \\ p \equiv a(q) \\ p \le X}} \log p - \frac{X}{\varphi(q)} \Bigg| \le C(A,\varepsilon) \frac{X}{(\log X)^A}$$

holds for any $A > 0$, $\varepsilon > 0$ and $X > 0$ $\quad \big[\Leftrightarrow EH(\vartheta)\big]$.

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Theorem (GPY 2005–2006–2009): *If $EH(\vartheta)$ is true for some $\vartheta > \frac{1}{2}$, then $d_n \le C(\vartheta)$ i.o. Furthermore $C(1) = 16$.*

Dickson's Conjecture (1904): *If $a_i n + b_i$ are linear forms with $a_i, b_i \in \mathbb{Z}$, $a_i > 0$, $\prod_{i=1}^{k}(a_i n + b_i)$ has no fixed prime divisor, then $\{a_i n + b_i\}_{i=1}^{k} \in \mathcal{P}^k$ for infinitely many n (i.o.).*

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Definition: A k-tuple $\mathcal{H}_k = \{h_i\}_{i=1}^{k}$, $0 \le h_1 < h_2 < \cdots < h_k$ is *admissible* if it covers $\nu_p < p$ residue classes $\bmod p$ for any prime p.

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Hardy–Littlewood's Conjecture (1923): *If $\mathcal{H}_k$ is admissible, then*

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$$\sum_{\substack{n<x \\ \{n+h_i\}\in\mathcal{P}^k}} 1 \sim \mathfrak{S}(\mathcal{H}_k)\frac{x}{\log^k x} \quad \left(\mathfrak{S}(\mathcal{H}_k) = \prod_p \left(1 - \frac{\nu_p}{p}\right)\left(1 - \frac{1}{p}\right)^{-k} > 0\right).$$

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Conjecture DHL (k): *If $\mathcal{H}_k$ is admissible, then $\{n + h_i\}_{i=1}^{k} \in \mathcal{P}^k$ i.o.*

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Theorem (GPY, 2005–2006–2009): *If* $EH(\vartheta)$ *is true,* $\vartheta > \frac{1}{2}$, *then* $\exists k_0 = C_1(\vartheta)$ *such that* $\mathrm{DHL}(k,2)$ *is true for any* $k \geq k_0$.

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Theorem (GPY, 2005–2006–2009): *If EH(ϑ) is true, $\vartheta > \frac{1}{2}$, then $\exists k_0 = C_1(\vartheta)$ such that $\mathrm{DHL}(k,2)$ is true for any $k \ge k_0$.*

Corollary: *If EH(ϑ) is true for some $\vartheta > \frac{1}{2}$, then $d_n < C_2(\vartheta)$ i.o.*

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Corollary: *If $EH(\vartheta)$ is true for some $\vartheta > \frac{1}{2}$, then $d_n < C_2(\vartheta)$ i.o.*

However, it suffices to show a conjecture weaker than $EH(\vartheta)$ for some $\vartheta > \frac{1}{2}$ and still obtain $\mathrm{DHL}(k_0, 2)$ and thus bounded gaps i.o.

Theorem: *Y. Motohashi – J. Pintz, A smoothed GPY-sieve, arXiv: math/0602599, Feb 27, 2006, Bull. London Math. Soc. 40 (2008), no. 2, 298–310 and www.renyi.hu/∼pintz, MR2414788 (2009d:1132).*

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It is sufficient to prove the analogue of $EH(\vartheta)$ with some $\vartheta > \frac{1}{2}$ for *smooth* moduli q (satisfying $p \mid q \to p < X^b$ with an arbitrary fixed $b > 0$) and for solutions a of the congruence $\prod_{i=1}^{k}(a + h_i) \equiv 0 \pmod{q}$ as residue classes $\bmod\ q$.

Y. Zhang's Theorem (2013, Ann. of Math., to appear). *$EH(\vartheta)$ is true for $\vartheta = \frac{1}{2} + \frac{1}{584}$ for smooth moduli and solutions of the congruence $\prod_{i=1}^{k}(a + h_i) \equiv 0 (\mathrm{mod}\ q)$.*

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Corollary 1: *$\mathrm{DHL}(k, 2)$ is true for $k \geq 3.5 \cdot 10^6$.*

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Corollary 1: $\mathrm{DHL}(k, 2)$ *is true for* $k \geq 3.5 \cdot 10^6$.
Corollary 2: $d_n = p_{n+1} - p_n < 7 \cdot 10^7$ *i.o.*

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Remark. 70 million is being improved to a few thousands (T. Tao's blog and Polymath project).

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Solutions: $k = 3$ K.F. Roth (1952–53)
$k = 4$ E. Szemerédi (1968–70)
k arbitrary: E. Szemerédi (1973–75) Abel prize 2012
H. Fürstenberg (1977) Wolf prize 2006/7
T. Gowers (1998) Fields medal 1998

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Methods (ergodic – Fürstenberg, harmonic analysis – Gowers, combinatorial – Szemerédi + number theoretical – Goldston–Yıldırım)

Erdős Conjecture (USD 3000): *If* $\sum_{a_i \in \mathcal{A}} 1/a_i = \infty$, *then* $\mathcal{A}$ *contains infinitely many* k*-term AP's for any* k.

4. ARITHMETIC PROGRESSIONS OF GENERALIZED TWIN PRIMES

Theorem 1 (J. P., arXiv 2013): *There exists an even $d < 7000$ with the following property. For every k there is a k-term AP of primes such that for each element p of the progression $p + d$ is also a prime, more exactly, the prime following p.*

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6. THE NORMALIZED VALUE DISTRIBUTION OF d_n

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Theorem 4 (J. P., arXiv 2013): *$\exists c$ (ineffective) such that $[0,c] \subset J$.*

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Erdős (1948) $\liminf_{n\to\infty} \frac{d_{n+1}}{d_n} < 1 < \limsup \frac{d_{n+1}}{d_n}$

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Theorem 5 (J. P., arXiv 2013): *Erdős's conjecture* (5) *is true, we have even*

(6) $$\liminf_{n\to\infty} \frac{d_{n+1}\log n}{d_n} < \infty, \quad \limsup_{n\to\infty} \frac{d_{n+1}}{d_n \log n} > 0$$

(7) $$\limsup_{n\to\infty} \frac{\min(d_{n-1}, d_{n+1})}{d_n(\log n)^c} = \infty \text{ with } c = 10^{-3}$$

8. CONJECTURES OF ERDŐS AND ERDŐS–MIRSKY ON CONSECUTIVE VALUES OF ARITHMETIC FUNCTIONS

Conjecture A: $d(n) = d(n+1)$ *i.o.* (Erdős–Mirsky 1952)

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Def: $\Omega(n)$ and $\omega(n)$ denote the number of prime divisors of n with ($\Omega(n)$) or without ($\omega(n)$) multiplicity.

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We conjecture that $2p+1 = p_1 p_2$ i.o. Then for these primes
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Erdős's conjectures were considered as difficult as the twin prime conjecture.

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In joint work with S. W. Graham, D. Goldston, C. Yıldırım
Theorem 6 (GGPY 2009): *Let q_n denote the sequence of E_2 numbers which have exactly two prime divisors. Then $q_{n+1} - q_n \leq 6$ i.o.*

Theorem 7 (GGPY): *For every* $B \geq 0$ $(B \in \mathbb{Z}^+)$ $\exists$ *inf. many n's with*

$$\omega(n) = \omega(n+1) = 4 + B,\ \Omega(n) = \Omega(n+1) = 5 + B, \quad (9)$$
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Theorem 8 (GGPY 2011, GGPY 2011):

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Theorem 9 (J. P. 2011): $\forall k$ $\exists$ *k-term AP of natural numbers n such that* (9) *is true. The same assertion holds for* (10) *and* (11).

9. SOME IDEAS OF PROOF BEHIND THEOREMS 6–8

The following Basic Theorem forms the basis for the proofs of Theorems 6–9.

BASIC THEOREM (S.W. Graham–Goldston–Pintz–Yıldırım): *If* $L_i(x) = a_i x + b_i$ $(i = 1, 2, 3,\ a_i, b_i \in \mathbb{Z},\ a_i > 0)$ *are three linear forms such that* $\prod_{i=1}^{3} L_i(x)$ *has no fixed prime divisor, then we have at least two indices* $i, j \in (1, 2, 3)$ *such that for any* C *and infinitely many* n

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Corollary: *Take* $\{n,\ n+2,\ n+6\} \Rightarrow q_{n+1} - q_n \le 6$ *i.o.*

Proof of (10) of Theorem 8 from the BASIC THEOREM
Let $L_1(m) = 6m + 1$, $L_2(m) = 8m + 1$, $L_3(m) = 9m + 1$.

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This is clearly admissible since $\prod_{i=1}^{3} L_i(0) \equiv 1 \pmod p$. We have
$4L_1 = 3L_2 + 1$, $3L_1 = 2L_3 + 1$, $9L_2 = 8L_1 + 1$.
Suppose, e.g., $L_1(n)$ and $L_2(n)$ are E_2-numbers i.o. If $x = 3L_2(n)$, $x + 1 = 4L_1(n)$, $n \not\equiv 1 (\mathrm{mod}\ 3)$, then $\omega(x) = \omega(x + 1)$ i.o.

10. SOME IDEAS BEHIND THE PROOF OF ZHANG'S THEOREM

The proof relies on three theorems:

(i) GPY (2005–6–9). If $\exists\vartheta > 1/2$ s.t. EH(ϑ) is true, i.e. for any $A, \varepsilon > 0$

$$\sum_{q \le X^{\vartheta-\varepsilon}} \max_{\substack{a \\ (a,q)=1}} \left| \sum_{p \le X,\ p \equiv a(q)} \log p - \frac{X}{\varphi(q)} \right| \le C(A,\varepsilon) \frac{X}{(\log X)^A}, \tag{13}$$

then DHL(k, 2) is true for $k \ge k_0 = C_3(\vartheta)$, i.e. we have for any admissible $\mathcal{H}_k$ at least two primes among $\{n + h_i\}_{i=1}^{k}$ i.o.

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(iii) ZHANG (to appear): (13) is true if restricted by (ii).

Ideas to prove (i) go back to Selberg and Heath-Brown. Since $n+\mathcal{H}_k$ contains just $\dfrac{k}{\log N}$ primes if $n \sim N$ ($n \in [N, 2N)$) on average, we look for an average which gives large weights a_n if $n+\mathcal{H}$ contains many primes. Let $\mathcal{P}_{\mathcal{H}}(n) = \prod_{i=1}^{k}(n+h_i)$.

1. $a_1(n) = \begin{cases} 1 & \text{if } \{n+h_i\}_{i=1}^{k} \in \mathcal{P}^k \text{ (tautology)} \\ 0 & \text{otherwise} \end{cases}$

2. $a_2(n) = \Lambda_k\big(P_{\mathcal{H}}(n)\big) = \sum_{d \mid P_{\mathcal{H}}(n)} \mu(d) \left(\log \dfrac{P_{\mathcal{H}}(n)}{d}\right)^k$

is a reformulation of $a_1(n)$ ($a_2(n) = a_1(n)$): we cannot evaluate $S(N) = \sum_{n \sim N} a_n$.

3. $a_3(n) = \Lambda_{k,R}(n) = \sum_{\substack{d \mid P_{\mathcal{H}}(n) \\ d \le R}} \mu(d) \log^k \frac{R}{d}$

(Selberg's idea). Problem: $a_3(n)$ may be negative.

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4. $a_4(n) = (a_3(n))^2$. First chanceful choice!
$S(N)$ can be evaluated; further if

$$\chi_{\mathcal{P}}(n) = \begin{cases} 1 & \text{if } n \in \mathcal{P} \\ 0 & \text{otherwise,} \end{cases} \text{ then } S^*(N) = \sum_{i=1}^{k} \sum_{n \sim N} a_n \chi_{\mathcal{P}}(n + h_i)$$

can be evaluated as well if $R \le N^{1/4 - o(1)}$.

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We obtain $\frac{S^*(N)}{S(N)} = \frac{1}{2} + o_k(1)$ primes "on average".

5. All attempts 1–4 simulate the full DHL(k) conjecture, i.e. to obtain k primes in a k-tuple i.o. (Dickson's conjecture). Let's be more modest. We are contented if we approximate DHL($k, 2$), i.e. if we have $k + \ell$ prime factors of $\prod_{i=1}^{k}(n + h_i)$ for some $\ell \le k - 2$.

$$(14) \quad a_5(n) = \Lambda^2_{k+\ell,R}(n) = \sum_{\substack{d \mid P_{\mathcal{H}}(n) \\ d \le R}} \mu(d) \left(\log \frac{R}{d}\right)^{k+\ell}, \ R \le N^{\frac{1}{4}-\varepsilon}.$$

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We obtain

$$\frac{S^*(N)}{S(N)} = 1 - O\left(\frac{\ell}{k}\right) - O\left(\frac{1}{\ell}\right) \tag{15}$$

primes on average over $n \sim N$ (unconditionally).

With some additional ideas this leads to the Small Gaps Conjecture, i.e. $\Delta = \liminf_{n\to\infty} \frac{d_n}{\log n} = 0$. However, conditionally, if $\vartheta > \frac{1}{2}$, EH(ϑ) is true, then

$$\frac{S^*(N)}{S(N)} = 2\vartheta\left(1 - O\left(\frac{\ell}{k}\right) - O\left(\frac{1}{\ell}\right)\right) > 1 \text{ (i)}. \tag{16}$$

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(ii) MOTOHASHI–PINTZ: If we can show EH(ϑ) for a $\vartheta > \frac{1}{2}$ for smooth moduli ($p \mid q \to p > q^b$) and instead of the worst residue class $\bmod q$ for solutions of the congruence $\prod_{i=1}^{k}(a + h_i) = 0$, then we obtain under the condition $b \geq C\ell/k$

$$\text{(17)} \qquad \frac{S^*(N)}{S(N)} = 2\vartheta\left(1 - O\left(\frac{\ell}{k}\right) - O\left(\frac{1}{\ell}\right) + O(e^{-kb/3})\right) > 1.$$

(iii) ZHANG: It is possible to show the above mentioned restricted improvement of the Bombieri–Vinogradov theorem using methods of Bombieri–Friedlander–Iwaniec, Weil, Friedlander–Iwaniec (with an appendix of Bombieri–Birch) which apply a technique based on the theory of Kloosterman sums. It turned out later that the most useful idea is in Fouvry–Iwaniec (1980) which proves the following theorem. For every $a \le X$

$$\sum_{q \le X^{11/21}} \Bigg| \sum_{\substack{n \equiv a \pmod q \\ n \le X \\ p \mid n \to p \le z}} 1 - \text{Exp. Main Term} \Bigg| \le C(A) \frac{X}{\log^A X}$$

where $z = X^{1/883}$, $A > 0$, $X > 0$ arbitrary.

ANALOGY: The moduli are here arbitrary (rigid) but the numbers n are well factorable. In case of prime gaps we have a "dual" problem. By the Motohashi–Pintz theorem we can factorise q arbitrarily, and while the primes seem to be rigid, they might be written in a multilinear form using Linník's or Heath-Brown's identity. Crucial role is still played by Friedlander–Iwaniec (1985): $a \leq X$, $d_3(n) = \sum_{n=n_1 n_2 n_3} 1$,

$$\sum_{q \leq X^{1/2+1/230}} \left| \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} d_3(n) - \text{Exp. Main Term} \right| \leq C(A) \frac{X}{\log^A X}.$$

Crucial idea behind the proof of Theorems 1–5 (apart from earlier mentioned results)

MAIN LEMMA (J. P. 2010): *The total sum of weights* $a_5(n)$ *for numbers for which at least one of the numbers* $n + h_i$ $(i = 1, 2, \ldots, k)$ *has a divisor* $< n^b$ *is negligible* $\left(< \varepsilon \sum_{n=N}^{2N} a_5(n)\right)$ *if* $b < \varepsilon c(k)$.

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Corollary (GPY 2010): *Given any $\eta > 0$ a positive proportion of primegaps d_n satisfy $d_n < \eta \log n$.*

Theorem 10. *If* $k \geq k_0$, $\mathcal{H} = \{h_i\}_{i=1}^k$ *is an admissible k-tuple, then for* $N > N_0(k)$ *the number of* $n \in [N, 2N)$ *for which* $\{n + h_i\}_{i=i}^k$ *contains at least two primes and almost primes in all other components with all prime factors* $> n^{c_1(k)}$ *is at least*

$$c_2(k) \frac{N}{\log^k N}$$

if $0 \leq h_i \ll \log N$.

"Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have no reason to believe that it is a mystery into which the mind will ever penetrate."

Leonhard Euler