Asymptotic Structure of Graphs with the Minimum Number of Triangles

Oleg Pikhurko

University of Warwick

Erdős Centennial Conference

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Barbie Benzer, Miriam and Debbie Golomb

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- Mantel 1906, Turán'41: $\max\{m : g(n,m) = 0\} = \lfloor \frac{n^2}{4} \rfloor$
- Rademacher'41: $g(n, \lfloor \frac{n^2}{4} \rfloor + 1) = \lfloor \frac{n}{2} \rfloor$

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- Moon-Moser'62, Nordhaus-Stewart'63, Bollobás'76...

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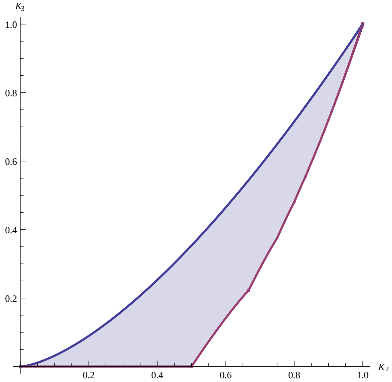

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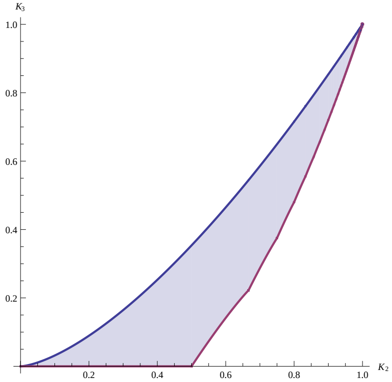

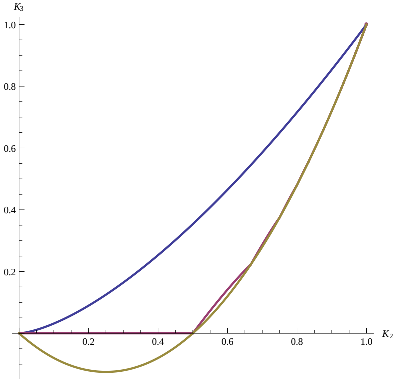

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  $\forall$ almost extremal G_n is $o(n^2)$-close to some H_n^a

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  - Close to H_n^a in cut-distance $\Rightarrow$ close in edit distance

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  - G_n has $\lessapprox cn$ triangles on almost every edge

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  - K_3^E: Density of rooted triangles

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$$-3!\,\phi_{\mathbf{0}}^{\mathbf{1}}(K_3^1) + 3\phi_0(K_3) = 3c\left(-2\phi_{\mathbf{0}}^{\mathbf{1}}(K_2^1) + 2a\right) \quad a.s.$$

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  - $\phi_{\mathbf{0}}^{\boldsymbol{E}}(K_3^E) \leq c$

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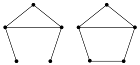
- Special graphs F_1 and F_2: 

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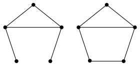
- Special graphs F_1 and F_2: 
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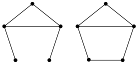
- Special graphs F_1 and F_2: 
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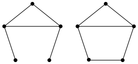
- Special graphs F_1 and F_2: 
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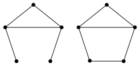
- Special graphs F_1 and F_2: 
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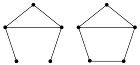
- Special graphs F_1 and F_2: 
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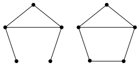
- Special graphs F_1 and F_2: 
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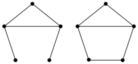
- Special graphs F_1 and F_2: 
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Thank you!

Photos: Math PUrview, Gil Kalai's blog & Erdős Lap Number ☺