Coloring a graph arising from a lacunary sequence, Diophantine approximation, and constructing a Kakeya set: Applications of the probabilistic method

Yuval Peres [1]

Based on joint works with

Wilhelm Schlag;

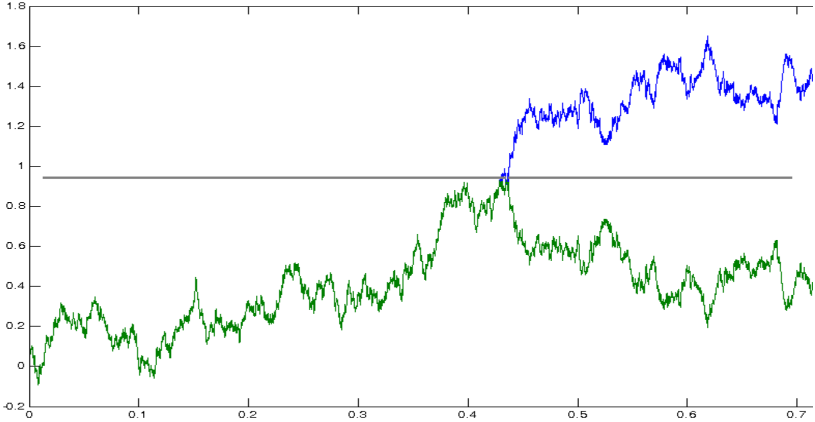
Y. Babichenko, R. Peretz, P. Sousi and P. Winkler

[1] Microsoft Research

- A. Dvoretzky, P. Erdős and S. Kakutani,
  Double points of paths of Brownian motion in *n*-space,
  Acta Sci. Math. Szeged,12 75–81, 1950.
- A. Dvoretzky, P. Erdős and S. Kakutani,
  Multiple points of paths of Brownian motion in the plane,
  Bull. Res. Council Israel, 3 364–371, 1954.
- A. Dvoretzky, P. Erdős, S. Kakutani and S. J. Taylor,
  Triple points of Brownian paths in 3-space,
  Proc. Cambridge Philos. Soc., 53, 856–862, 1957.
- A. Dvoretzky, P. Erdős and S. Kakutani,
  Points of multiplicity c of plane Brownian paths,
  Bull. Res. Council Israel 7, 175–180, 1958.
- A. Dvoretzky, P. Erdős and S. Kakutani,
  Nonincrease everywhere of the Brownian motion process,
  Proc. 4th Berkeley Sympos. II 103–116, 1961.

Fine properties of Brownian paths, following Paul Lévy and continued by S.J. Taylor, J.F. Le Gall and many others.

Does Brownian motion have points of increase?



Nonincrease everywhere of the Brownian motion process; Dvoretzky, Erdős and Kakutani (1961).

From the review by J. Lamperti:

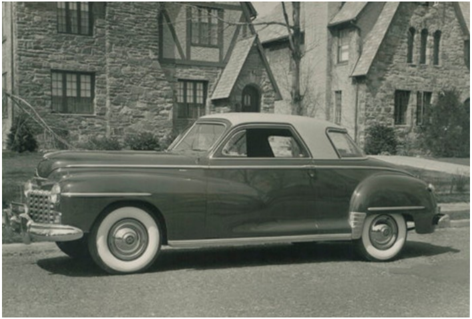
> *Let X be the standard Brownian motion in one dimension. It is well-known that, with probability 1, a path of this process is nowhere differentiable; the present paper establishes the more delicate fact that almost all Brownian paths have no points of increase. The proof is quite intricate . . .*

Simple proofs:

- K. Burdzy (Ann. Probab. 1990)
- Y.P. (Israel J. Math. 1996).

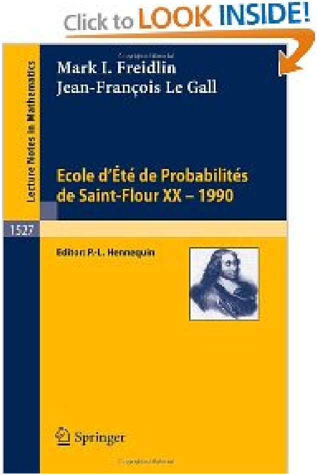
Kakutani's car

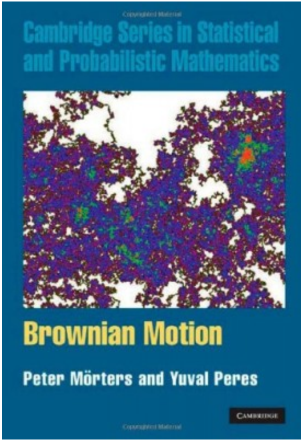
According to S. Kakutani (1990), DEK first found a "proof" that points of increase do exist, by a fancy version of the reflection principle . . .



Some echoes of 3AM can be found in the original paper . . .

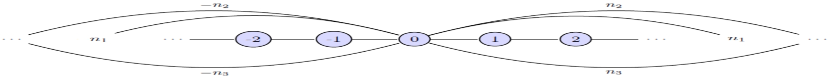
For more information on Brownian sample paths:





The Cayley graph determined by a sequence $\mathcal{S} = \{n_k\}$

Define a graph $\mathcal{G}_{\mathcal{S}}$ with vertex set $\mathbb{Z}$, where the pair $\{n, m\}$ is an edge iff $|n - m| \in \mathcal{S}$.



Example: $n_k = k^d$ where $2 < d \in \mathbb{N}$.
• $\mathcal{G}_{\mathcal{S}}$ has no triangles by FLT
• Furstenberg (1977) and Sárközy (1978) showed that $\forall A \subset \mathbb{Z}$ of positive upper density, $\exists x, y \in A$ and $k \in \mathbb{N}$ such that $x - y = k^d$.
• Thus every independent set in $\mathcal{G}_{\mathcal{S}}$ has zero density $\Rightarrow$
• The chromatic number $\chi(\mathcal{G}_{\mathcal{S}}) = \infty$.

Two problems of Erdős on lacunary sequences

• The chromatic number $\chi(\mathcal{G})$ of a graph $\mathcal{G}$ is the minimal number of colors in a proper vertex coloring (neighbors assigned distinct colors.)

Problem A (Erdős, 1987)

Fix $\varepsilon > 0$ and suppose $\mathcal{S} = \{n_j\}_{j=1}^{\infty}$ is a **lacunary** sequence of positive integers, where $n_{j+1} > (1+\varepsilon)n_j$ for all $j \geq 1$. Is the chromatic number $\chi(\mathcal{G}_{\mathcal{S}})$ necessarily finite?

Problem B (Erdős, 1975)

Let $\varepsilon > 0$ and $\mathcal{S}$ be as in Problem A. Is there a number $\theta \in (0,1)$ so that the sequence $\{n_j\theta\}_{j=1}^{\infty}$ is not dense modulo 1?

The relation between Problem A and Problem B was discovered by Katznelson in 1987, and published in 2001.

Relation between Problem A and Problem B

• Let $\delta > 0$ and $\theta \in (0, 1)$ be such that $\inf_j \|\theta n_j\| > \delta$, where $\|\cdot\|$ is distance to the closest integer.

• Partition $\mathbb{T} = [0, 1)$ into $k = \lceil \delta^{-1} \rceil$ disjoint intervals $I_1, \ldots, I_k$ of length $\frac{1}{k} \leq \delta$.

• Let $\mathcal{G}$ be the graph from Problem A and assign the vertex $n \in \mathbb{Z}$ the color j iff $n\theta \in I_j \pmod 1$.

• Any two vertices connected by an edge must have different colors. Therefore, $\chi(\mathcal{G}) \leq k = \lceil \delta^{-1} \rceil$.

• Problem B was solved by Pollington (1979), de Mathan (1980) and Katznelson (2001); • As noted by Moshchevitin (2010), problem B

was already raised and solved in 1926 by Khinchin, but this was forgotten . . .

• Khinchin (1926) and Katznelson (2001) showed that there exists a θ such that

$$\inf_{j\geq 1} \|\theta n_j\| > c\varepsilon^2 |\log \varepsilon|^{-1}.$$

Main Result

Theorem (P., Schlag; Bull. London Math. Soc. 42 (2010))

Suppose $\mathcal{S} = \{n_j\}$ satisfies $n_{j+1}/n_j \geq 1 + \varepsilon$, where $0 < \varepsilon < 1/4$. Then there exists $\theta \in (0, 1)$ such that

$$\inf_{j \geq 1} \|\theta n_j\| > c\varepsilon |\log \varepsilon|^{-1}, \tag{1}$$

where $c > 0$ is a universal constant. Therefore, the graph $\mathcal{G} = \mathcal{G}_{\mathcal{S}}$ described in Problem A satisfies $\chi(\mathcal{G}) \leq c^{-1}|\log \varepsilon|/\varepsilon$.

• Up to the $|\log \varepsilon|^{-1}$ factor, (1) is optimal. Indeed, let $n_j = j$ for $j = 1, 2, \ldots, \lfloor \varepsilon^{-1} \rfloor$ and continue this as a lacunary sequence with ratio $1 + \varepsilon$. In this case $\chi(\mathcal{G}) > \lfloor \varepsilon^{-1} \rfloor$.

Rotation orbits sampled along a lacunary sequence

The following quantitative result on Problem B extends the previous theorem.

Theorem (P., Schlag 2010)

Suppose $\mathcal{S} = \{n_j\}$ *satisfies* $n_{j+1}/n_j \geq 1 + \varepsilon$ *for all* j. *Define*

$$E_j = \left\{ \theta \in \mathbb{T} \, : \, \|n_j \theta\| < \frac{c_0 \varepsilon}{|\log_2 \varepsilon|} \right\} \tag{2}$$

for $j \geq 1$. *If* $240\, c_0 \leq 1$, *then*

$$\bigcap_{j=1}^{\infty} E_j^c \neq \emptyset. \tag{3}$$

Proof ingredient: Lovász local lemma

Lemma

Let $\{A_j\}_{j=1}^N$ *be events in a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and let* $\{x_j\}_{j=1}^N$ *be a sequence of numbers in* $(0,1)$. *Assume that for every* $i \leq N$, *there is an integer* $0 \leq m(i) < i$ *so that*

$$\mathbb{P}\Big(A_i \Big| \bigcap_{j<m(i)} A_j^c\Big) \leq x_i \prod_{j=m(i)}^{i-1} (1-x_j). \tag{4}$$

Then for any integer $n \in [1, N]$, *we have*

$$\mathbb{P}\Big(\bigcap_{i=1}^n A_i^c\Big) \geq \prod_{\ell=1}^n (1-x_\ell). \tag{5}$$

The lemma is applied to Lebesgue measure in $[0,1]$ and to sets $\{A_j\}$, where A_j is the union of all binary intervals of length $\dfrac{c_0\varepsilon}{n_j|\log_2 \varepsilon|}$ that intersect E_j.

MR2770060 Y. Bugeaud and N. Moshchevitin (2011)
Badly approximable numbers and Littlewood-type problems.
Math. Proc. Cambridge Philos. Soc. 150, 215-226.
From Math Reviews:

> *The Littlewood conjecture states that, for any given pair (α, β) of real numbers, we have* $\inf_{q\geq 1} q \cdot \|q\alpha\| \cdot \|q\beta\| = 0$*, where $\|\cdot\|$ denotes the distance to the nearest integer. The authors prove, with a method introduced by Y. Peres and W. Schlag, that the set of pairs $(\alpha, \beta) \in \mathbb{R}^2$ such that*
>
> $$\varliminf_{q\to+\infty} q \cdot (\log q)^2 \cdot \|q\alpha\| \cdot \|q\beta\| > 0$$
>
> *has full Hausdorff dimension in* $\mathbf{R}^2$.

Kakeya sets – History

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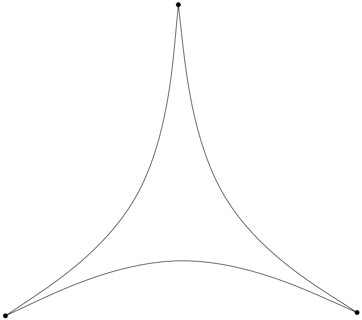
Kakeya's question (1917): Is the three-pointed deltoid shape a Kakeya set of minimal area?



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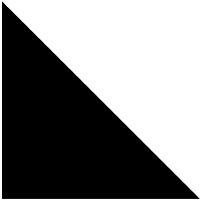

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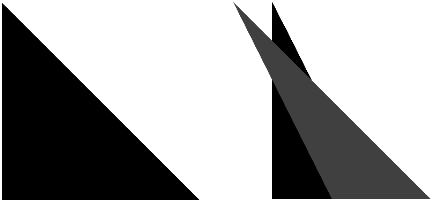

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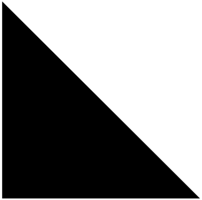

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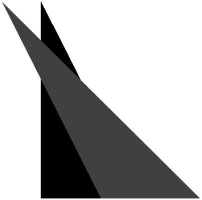

$n = 2$

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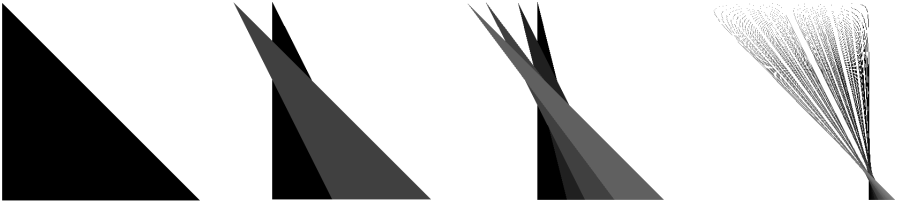

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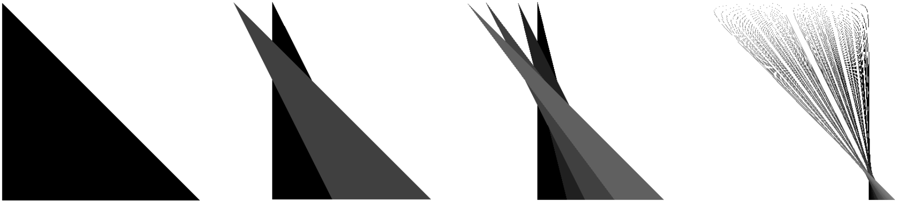

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(Figures due to Terry Tao)

New connection to game theory and probability

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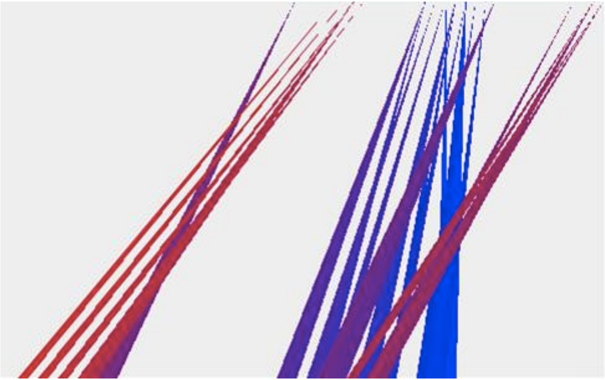

A. S. Besicovitch.
On Kakeya's problem and a similar one.
Math. Z., 27(1):312–320, 1928.

Roy O. Davies.
Some remarks on the Kakeya problem.
Proc. Cambridge Philos. Soc., 69:417–421, 1971.

Micah Adler, Harald Räcke, Naveen Sivadasan, Christian Sohler, and Berthold Vöcking.
Randomized pursuit-evasion in graphs.
Combin. Probab. Comput., 12(3):225–244, 2003.

Yakov Babichenko, Yuval Peres, Ron Peretz, Perla Sousi, and Peter Winkler.
Hunter, Cauchy Rabbit and Optimal Kakeya Sets.
Transactions AMS, to appear; arXiv:1207.6389

Definition of the game G_n

Two players

Two players

Two players

Hunter

Definition of the game G_n

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Hunter

Two players

Hunter

Rabbit

Two players

Where?

Hunter

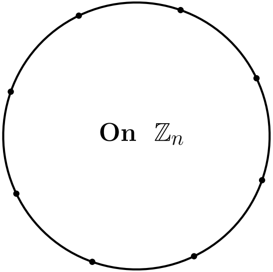
Rabbit

Two players

Hunter

Rabbit

Where?



When?

When?


Hi-res JPG file download - Resolution 5000x3750 px - www.psdgraphics.com

When?

At night – they cannot see each other....

Definition of the game G_n

Rules

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Examples of strategies

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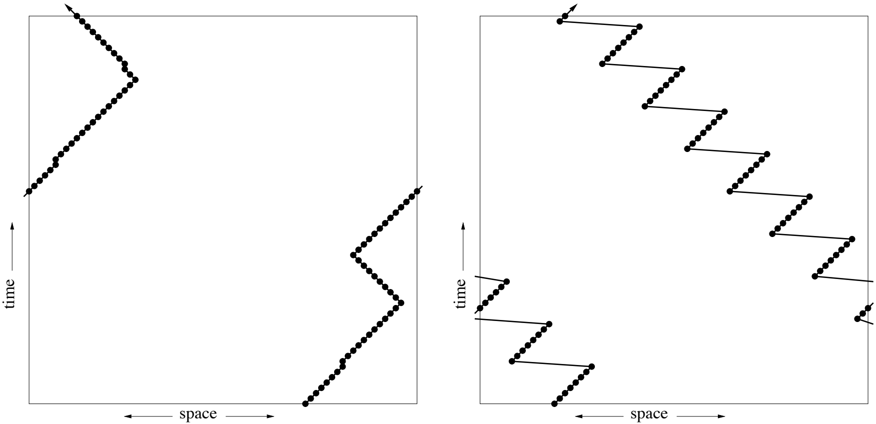
  Rabbit counter-strategy: From a random starting node, the rabbit walks $\sqrt{n}$ steps to the right, then jumps $2\sqrt{n}$ to the left, and repeats. The probability of capture in n steps is $\asymp n^{-1/2}$, so mean capture time is $n^{3/2}$.

Zig-Zag hunter strategy



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Use second moment method – calculate first and second moments of K_n.

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$H_t = \lceil an + bt \rceil \mod n$

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This means that given the **rabbit and hunter** collided, we want them to collide "a lot". The hunter can only move to neighbours or stay put.

So the **rabbit** should also choose a distribution for the jumps that favors short distances, yet grows linearly in time. This suggests a ***Cauchy random walk***.

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Intuition: If $X_1, \ldots$ are i.i.d. Cauchy random variables, i.e. with density $(\pi(1+x^2))^{-1}$, then $X_1 + \ldots + X_n$ is spread over $(-n, n)$ and with roughly **uniform distribution**.

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This is what we want- **But** in the discrete setting...

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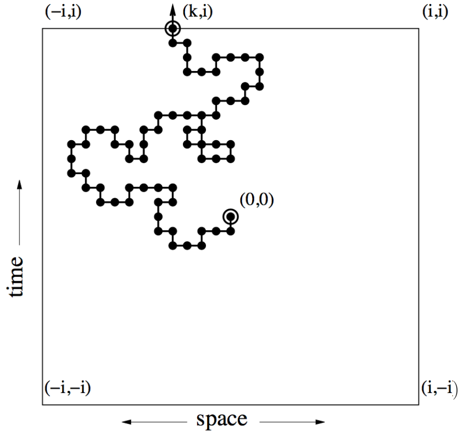

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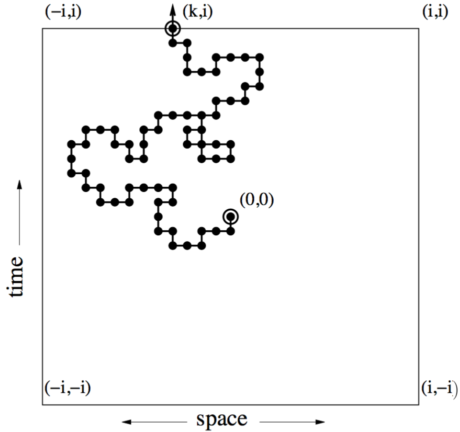

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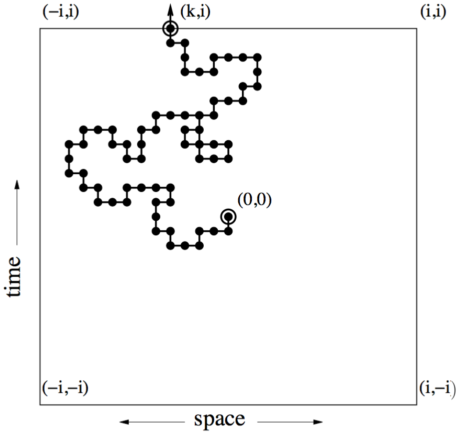

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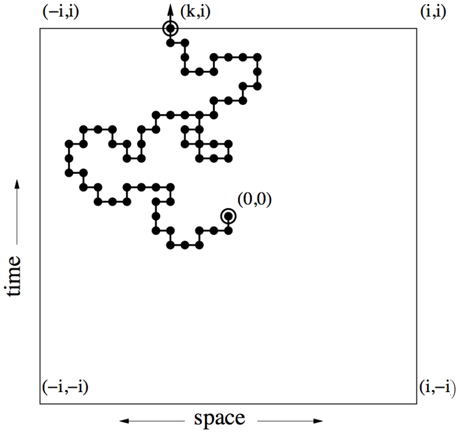
- Thus the hitting probability at $(0, i)$ is at least $1/(8i + 4)$.



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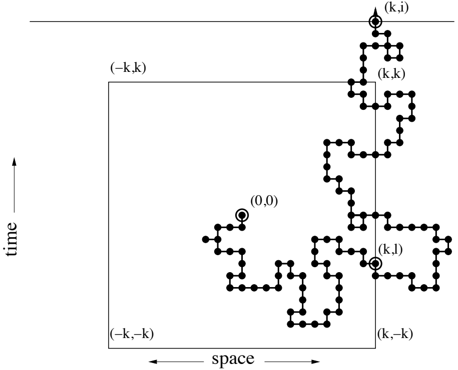

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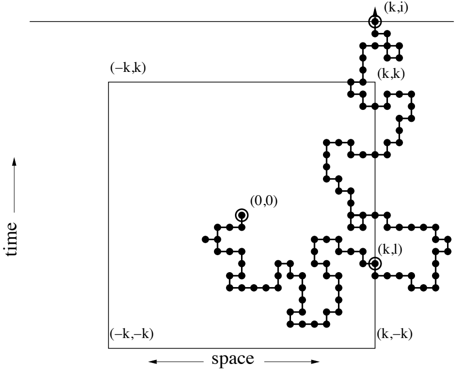

- Suppose $0 < k < i$.
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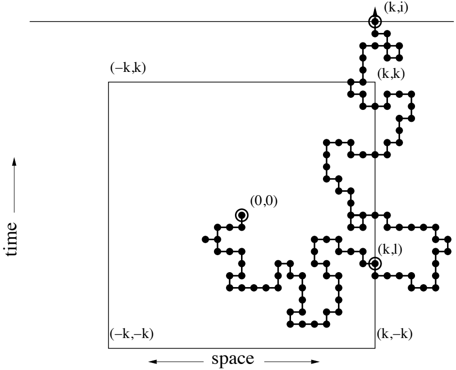

- Suppose $0 < k < i$.
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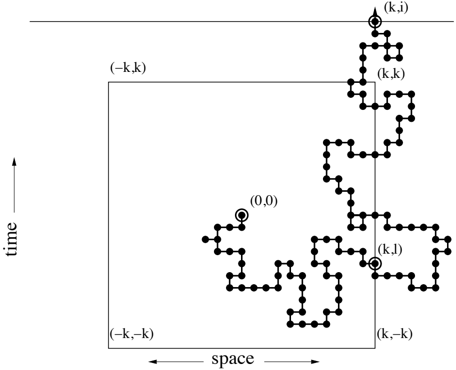
- Repeating the previous argument, the hitting probability at (k, i) is at least c/i.



Rabbit and Hunter construct Kakeya sets

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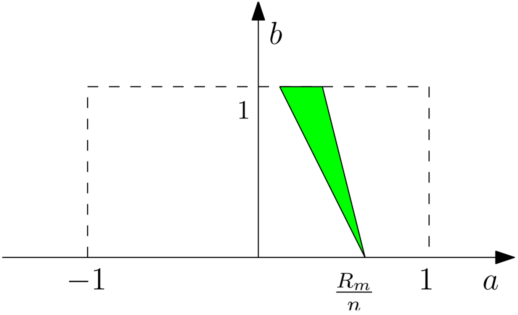

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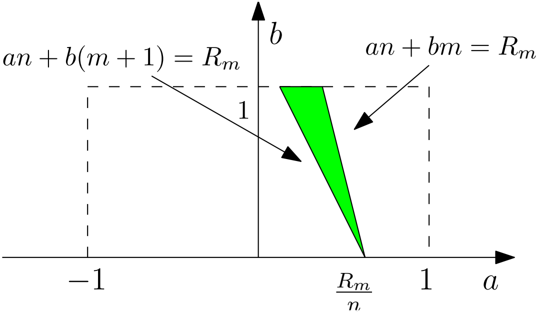

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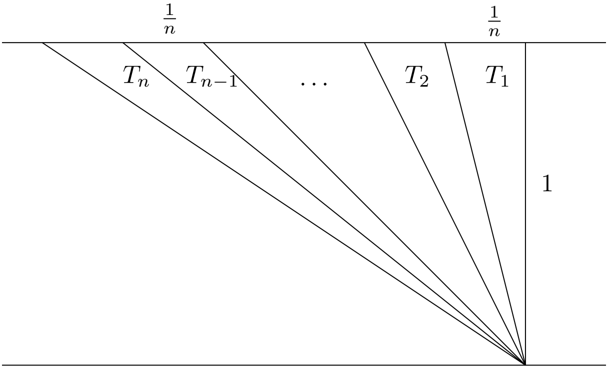
Hence the probability of collision in $[0, n)$ is half the area of the union of all such triangles, which are translates of



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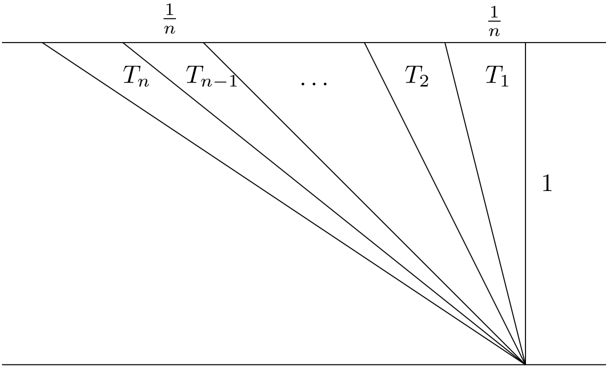

In these triangles we can find a unit segment in all directions that have an angle in $[0, \pi/4]$

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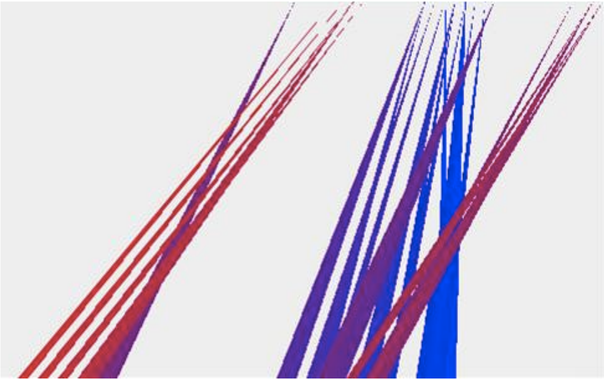

Simulation generated with $n = 32$

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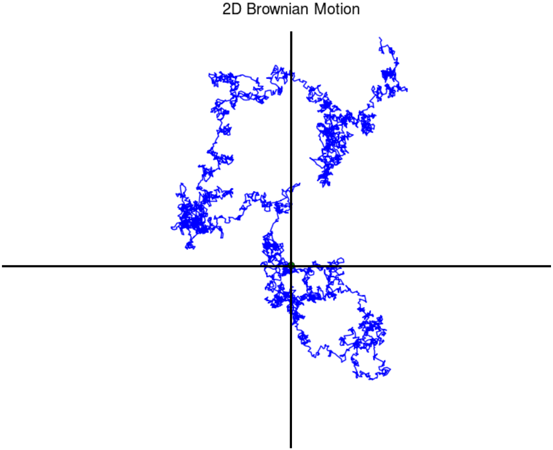

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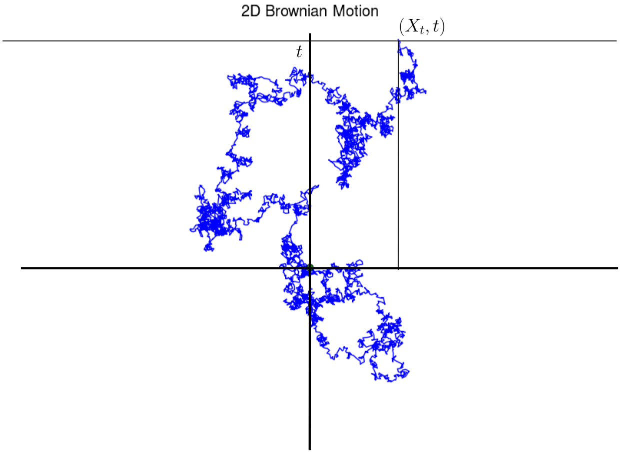

$X_{t+s} - X_t$ has the same law as tX_1 and X_1 has the Cauchy distribution (*density given by* $(\pi(1+x^2))^{-1}$).

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Λ is a quarter of a Kakeya set – it contains all directions from 0 up to 45° degrees. *Take four rotated copies of Λ to obtain a Kakeya set.*

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Let $(X_t)_t$ be a **Cauchy process**. Set

$$\Lambda = \{(a, X_t + at) : a, t \in [0, 1]\}.$$

Λ is a quarter of a Kakeya set – it contains all directions from 0 up to 45° degrees. *Take four rotated copies of Λ to obtain a Kakeya set.*

Λ is an optimal Kakeya set!

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$\text{Leb}(\Lambda) = 0$ and **most importantly** the ε-neighbourhood satisfies almost surely

$$\text{Leb}(\Lambda(\varepsilon)) \asymp \frac{1}{|\log \varepsilon|}$$

Kakeya sets – Open problems

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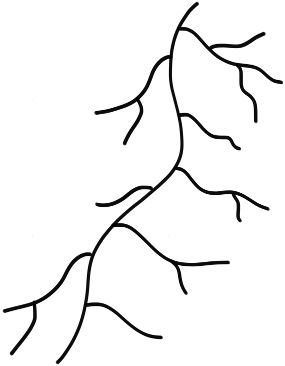

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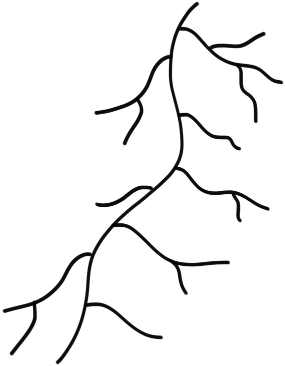

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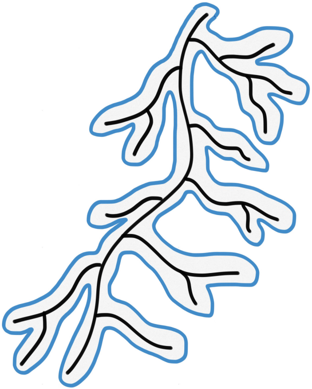

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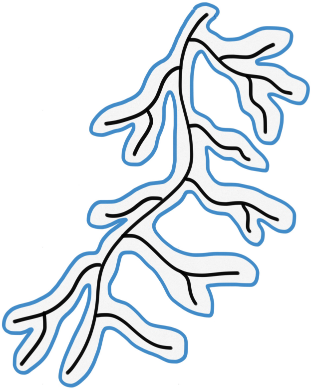

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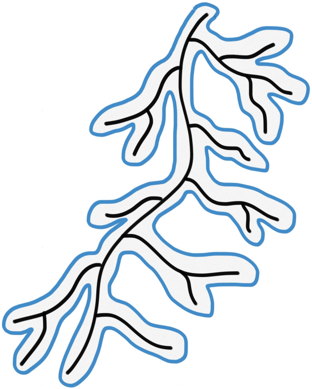
- This is a closed path of length $2n - 2$.
- The hunter can now employ his previous strategy on this path. This will give $O(n \log n)$ capture time.



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Open Question: If the hunter and rabbit both walk on the same graph, is the *expected capture time* $O(n)$?