Hamilton decompositions of regular expanders: proof of Kelly's conjecture for large tournaments

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Joint work with
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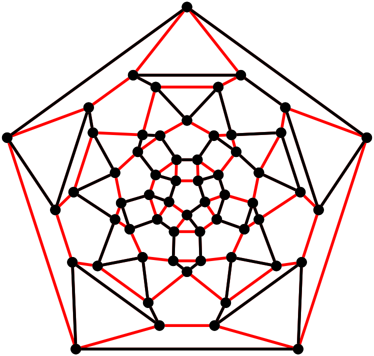
Hamilton decomposition of G
= set of edge-disjoint Hamilton cycles covering all edges of G

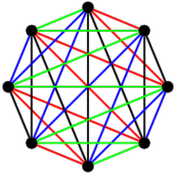


Which graphs/digraphs have Hamilton decompositions?
Very few general conditions known

Theorem (Walecki, 1892)

Complete graph K_n has a Hamilton decomposition $\Leftrightarrow$ n odd

Construction: find Hamilton path decomposition for K_{n-1}



then add extra vertex and close paths into Hamilton cycles

Theorem (Tillson, 1980)

Complete digraph K_n has a Hamilton decomposition $\Leftrightarrow$ $n \neq 4, 6$

Hamilton decompositions of tournaments

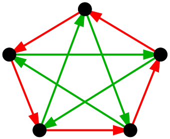
tournament: orientation of a complete graph

regular tournament: every vertex has same in- and outdegree

Conjecture (Kelly, 1968)

Every regular tournament has a Hamilton decomposition.

Decomposition of regular tournament into 2 Hamilton cycles



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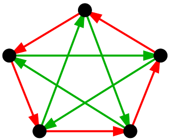

- even finding 2 edge-disjoint Hamilton cycles is not easy (Jackson 1981, Zhang 1980)
- further partial results due to Thomassen (1979, 1982), Alspach et al. (1990), Häggkvist (1993), Häggkvist & Thomason (1997), Bang-Jensen & Yeo (2004) ...

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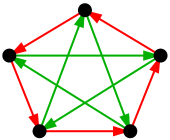

Theorem (Kühn, Osthus & Treglown, 2010)

Approximate version of Kelly's conjecture:
set of edge-disjoint Hamilton cycles covering almost all edges

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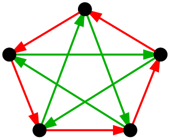


Theorem (Kühn & Osthus, 2013)

Every large regular tournament has a Hamilton decomposition.

Hamilton decompositions of oriented graphs

Result extends far beyond tournaments:

Theorem (Kühn, Osthus 2012^+)

Every large regular oriented graph of degree at least $\frac{3n+\varepsilon n}{8}$ has a Hamilton decomposition.

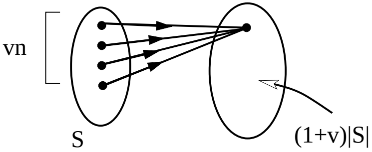
'$3n/8$' is a natural barrier
as this degree is needed to force single Hamilton cycle

Theorem (Keevash, Kühn & Osthus, 2009)

Every large oriented graph with $\delta^+, \delta^- \geq \frac{3n-4}{8}$ has a Hamilton cycle.

Robust outexpanders: main result

digraph G on n vertices is a **(ν, τ)-robust outexpander**
$\Leftrightarrow$ for every vertex set S with $\tau n \leq |S| \leq (1 - \tau)n$ there are $(1 + \nu)|S|$ vertices with νn inneighbours in S



Theorem (Kühn & Osthus 2013)

Suppose $1/n \ll \nu \ll \tau \ll \alpha$. Then every αn-regular (ν, τ)-robust outexpander on n vertices has a Hamilton decomposition.

Robust outexpanders: main result

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Digraph classes which are robust outexpanders:

(1) oriented graphs G with $\delta^+(G), \delta^-(G) \geq 3n/8 + o(n)$

(2) digraphs G with $\delta^+(G), \delta^-(G) \geq n/2 + o(n)$

(3) dense quasi-random digraphs

$\Rightarrow$ such digraphs have Hamilton decompositions if they are regular

So (3) generalizes result of Alspach, Bryant & Dyer (2012) on Hamilton decompositions of Paley graphs (for large n)

Finding Hamilton decompositions

Theorem (Kühn & Osthus 2013)

Every large regular tournament has a Hamilton decomposition.

Crucial notion: H is robustly decomposable if:
for any H' which is regular and sparse compared to H
$H \cup H'$ has a Hamilton decomposition

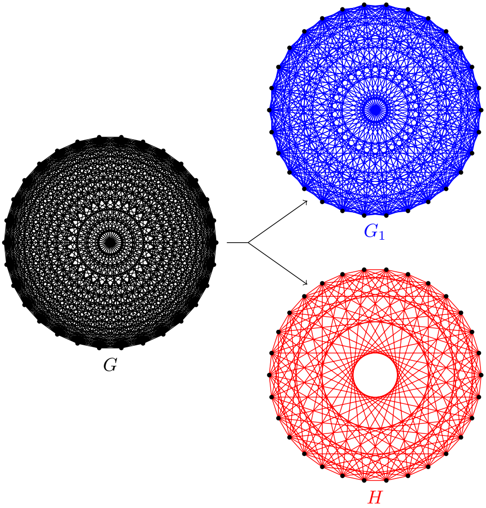
- Far from clear whether such H exists!!
- Will use this in combination with approx. result

Theorem (Kühn, Osthus & Treglown, 2010)

Every regular tournament G contains a set of edge-disjoint Hamilton cycles covering almost all the edges.

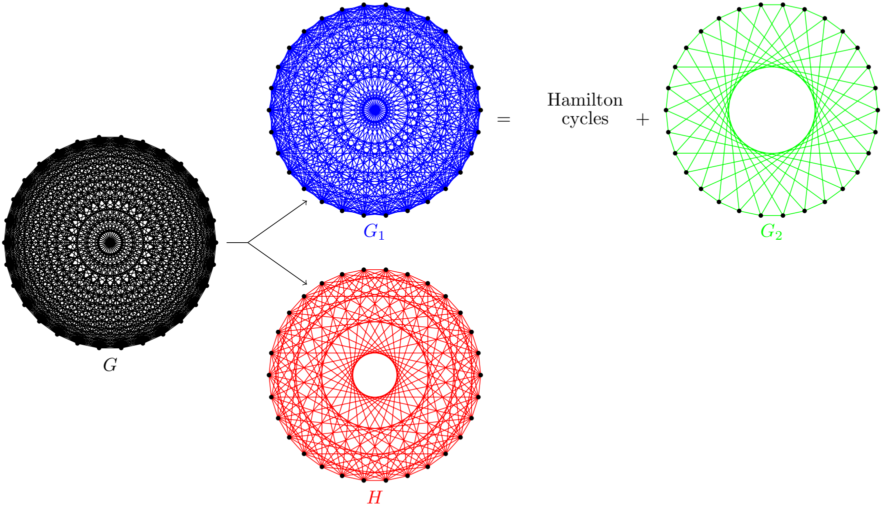
(generalized to regular outexpanders by Osthus & Staden 2013)

Finding Hamilton decompositions



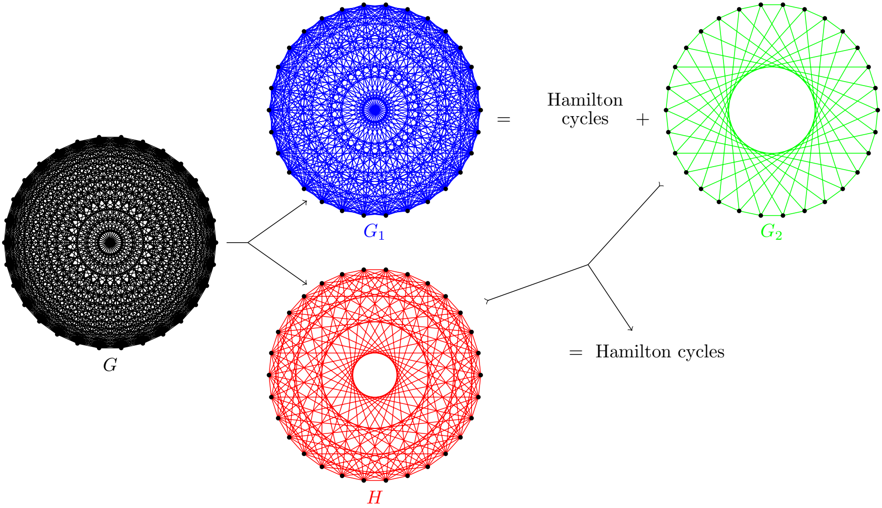
Remove a sparse regular **robustly decomposable** H to obtain G_1.

Finding Hamilton decompositions



Find an approximate decomposition of G_1. Call the leftover G_2.

Finding Hamilton decompositions



H robustly decomposable

$\Rightarrow$ there is a Hamilton decomposition of $G_2 \cup H$.

Applications of main result on robust outexpanders

Theorem (Kühn & Osthus 2013)

Suppose $1/n \ll \nu \ll \tau \ll \alpha$. *Then every* αn*-regular* (ν, τ)*-robust outexpander on* n *vertices has a Hamilton decomposition.*

Many applications:

- conjecture of Erdős from 1981 on random tournaments
- crucial ingredient for proof of the 1-factorization conjecture (1950's) and Hamilton decomposition conjecture (1970)
- conjecture of Nash-Williams from 1970's
- solves a problem on domination ratio for TSP tours by Glover & Punnen as well as Alon, Gutin & Krivelevich
- solves dense case of a conjecture of Frieze and Krivelevich on packing Hamilton cycles in random graphs
- ...

An application to random tournaments

Conjecture (Erdős, 1981)

Let T be a random tournament. Then a.a.s. T contains $\min\{\delta^+(T), \delta^-(T)\}$ edge-disjoint Hamilton cycles.

Note a random tournament is likely to be almost (but not completely!) regular.

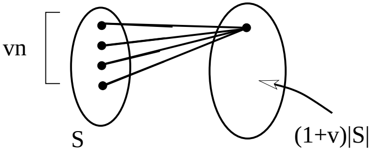
Proof of conjecture:

- Let $c = \min\{\delta^+(T), \delta^-(T)\}$. Then a.a.s. $c \sim n/2$.
- Can show T contains a c-regular oriented graph T'. (Find T' using the Max-flow-Min-cut theorem)
- $\Rightarrow$ T' is a robust outexpander
- $\Rightarrow$ T' has a Hamilton decomposition

Undirected robust expanders

Can deduce a version of main result for undirected graphs:

graph G on n vertices is a (ν, τ)-robust expander
$\Leftrightarrow$ for every vertex set S with $\tau n \leq |S| \leq (1-\tau)n$ there are $(1+\nu)|S|$ vertices with νn neighbours in S



Theorem (Kühn & Osthus 2013)

Every large even-regular robust expander G of linear degree has a Hamilton decomposition.

Undirected robust expanders

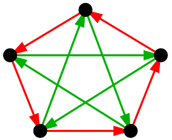
Theorem (Kühn & Osthus 2013)

Every large even-regular robust expander G of linear degree has a Hamilton decomposition.

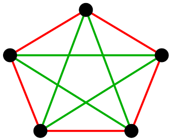
Proof strategy:

• Find orientation G_{orient} of G so that G_{orient} is a regular robust outexpander. Do this by choosing a random orientation and change directions of some edges to make it regular.

• Can apply main result on digraphs to obtain (directed) Hamilton decomposition of G_{orient}, which corresponds to (undirected) Hamilton decomposition of G.



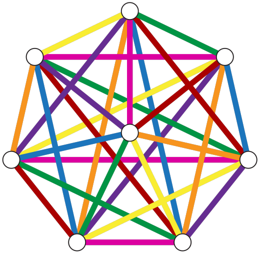
$\Rightarrow$



1-factorization conjecture

1-factor (or perfect matching) of G
$=$ set of disjoint edges covering all vertices of G
1-factorization of G
$=$ set of edge-disjoint 1-factors covering all edges of G

D-regular graph G has 1-factorization
$\iff$ G has an edge-colouring with D colours



A 1-factorization of the complete graph K_8 on 8 vertices

1-factorization conjecture

1-factorization conjecture (Dirac 1950's)

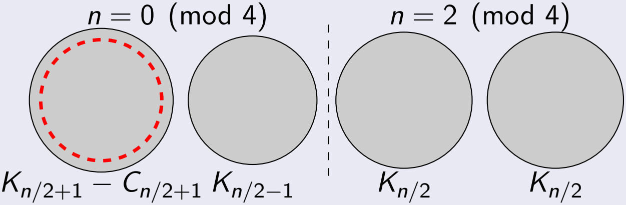
Every D-regular graph G on an even number n of vertices with $D \geq 2\lceil n/4 \rceil - 1$ has a 1-factorization.

Explicitly,

$$D \geq \begin{cases} n/2 - 1 & \text{if } n = 0 \pmod 4, \\ n/2 & \text{if } n = 2 \pmod 4. \end{cases}$$

Extremal examples

Odd component contains no 1-factor.



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- True for $D = n - 1$, i.e. complete graphs.
- Chetwynd and Hilton (1989), and independently Niessen and Volkmann (1990), for $D \geq (\sqrt{7} - 1)n/2 \approx 0.82n$.
- Perkovic and Reed (1997) for $D \geq (1/2 + \varepsilon)n$ with $\varepsilon > 0$.
- Vaughan (2013) : an approximate multigraph version.

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Theorem (Csaba, Kühn, Lo, Osthus, Treglown 2013$^+$)

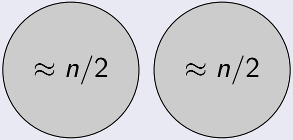
1-factorization conjecture holds for sufficiently large n.

Hamilton decomposition conjecture (Nash-Williams 1970)

Every D-regular graph on n vertices with $D \geq \lfloor n/2 \rfloor$ has a decomposition into Hamilton cycles and at most one perfect matching.

Extremal examples

No disconnected graph contains a Hamilton cycle.



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- Nash-Williams (1969), $D \geq \lfloor n/2 \rfloor$ guarantees Hamilton cycle.
- Jackson (1979), $D/2 - n/6$ edge-disjoint Hamilton cycles
- Christofides, Kühn and Osthus (2012) $D \geq n/2 + \varepsilon n$ guarantees an almost Hamilton decomposition.

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Theorem (Csaba, Kühn, Lo, Osthus, Treglown 2013^{+})

Hamilton decomposition conjecture holds for sufficiently large n.

Extremal structure

G is ε-close to H if G can be transformed to H by adding/removing at most εn^2 edges.

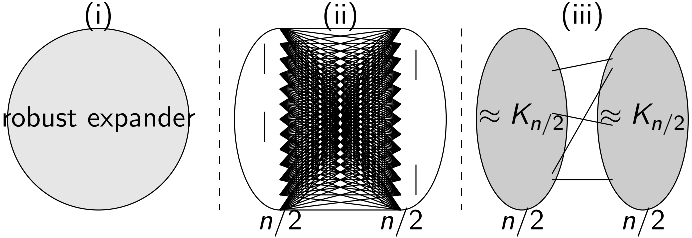
Lemma

Let G be a D-regular graph on n vertices and $D \geq n/2 - 1$. Then either

(i) *G is a robust expander;* ✓

(ii) *G is ε-close to complete bipartite graph $K_{n/2,n/2}$;*

(iii) *G is ε-close to union of two complete graphs $K_{n/2}$.*



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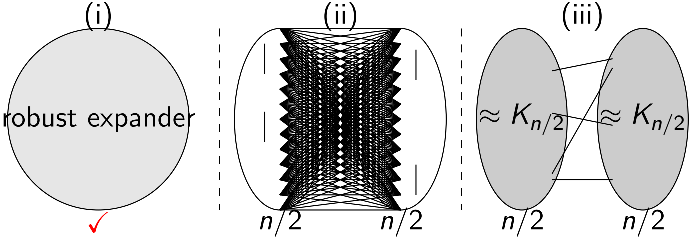

Edge-disjoint Hamilton cycles in graphs of large mindegree

Theorem (Dirac, 1952)

Suppose that G is a graph on $n \geq 3$ vertices with minimum vertex degree $\delta \geq n/2$. Then G has a Hamilton cycle.

Minimum degree condition is best possible.

Theorem (Nash-Williams, 1971)

Suppose that G is a graph on $n \geq 3$ vertices with minimum vertex degree $\delta \geq n/2$. Then G has a at least $5n/224$ edge-disjoint Hamilton cycles.

Conjecture (Nash-Williams):

can improve this to $n/4$ (clearly best possible)

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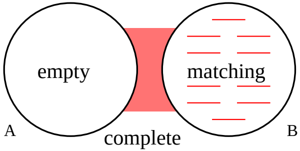
Babai: can't do better than $\approx n/8$

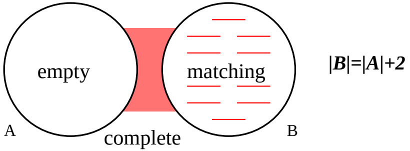
Babai's construction:



$|B| = |A| + 2$

Every Hamilton cycles contains at least 2 edges from B
$\Rightarrow$ G has at most $|B|/4$ edge-disjoint Hamilton cycles
$\Rightarrow$ G has $\leq (n+2)/8$ edge-disjoint Hamilton cycles

Babai's construction:



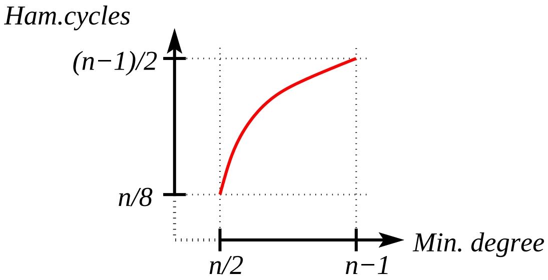
Every Hamilton cycles contains at least 2 edges from B
$\Rightarrow G$ has at most $|B|/4$ edge-disjoint Hamilton cycles
$\Rightarrow G$ has $\leq (n+2)/8$ edge-disjoint Hamilton cycles

Theorem (Csaba, Kühn, Lapinskas, Lo, Osthus, Treglown 2013$^+$)

If n is sufficiently large then this is the correct bound.

More generally:

determined the number of edge-disjoint Hamilton cycles which are guaranteed in a graph G of given minimum degree



extremal construction similar to Babai's example

Open problems

Have seen that robust expansion arises in many settings.

- Using Szemerédi's regularity lemma, can decide in polynomial time whether G is a robust expander.
- Robust expansion is a generalization of quasi-randomness.
- Quasi-randomness can be characterized by eigenvalues.

Question

Is there an algebraic characterization of robust expansion?