Low Tree-Depth Decompositions

Sparsity, Dualities, Logic, and Limits

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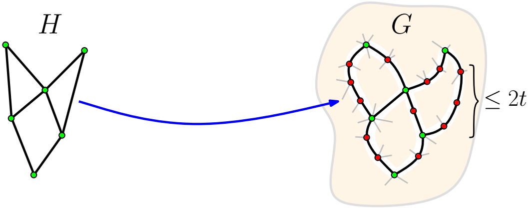
Outline

- Taxonomy of Graph Classes
- Low Tree-depth Decompositions
- Some Applications
- Structural Limits

Topological resolution of a class $\mathcal{C}$

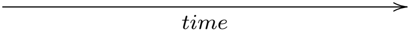
Shallow topological minors at depth t:



$$\mathcal{C} \,\widetilde{\nabla}\, t = \{H : \text{some} \leq 2t\text{-subdivision of } H \text{ is a subgraph of some } G \in \mathcal{C}\}.$$

Topological resolution:

$$\mathcal{C} \subseteq \mathcal{C} \,\widetilde{\nabla}\, 0 \subseteq \mathcal{C} \,\widetilde{\nabla}\, 1 \subseteq \dots \subseteq \mathcal{C} \,\widetilde{\nabla}\, t \subseteq \dots \subseteq \mathcal{C} \,\widetilde{\nabla}\, \infty$$



Taxonomy of Classes

A class $\mathcal{C}$ is *nowhere dense* if

$$\forall t \in \mathbb{N} : \quad \omega(\mathcal{C} \,\widetilde{\nabla}\, t) < \infty$$

$\ldots$ otherwise $\mathcal{C}$ is *somewhere dense*

$\mathcal{C}$ has *bounded expansion* if

$$\forall t \in \mathbb{N} : \quad \overline{\mathrm{d}}(\mathcal{C} \,\widetilde{\nabla}\, t) < \infty$$

Remark: bounded expansion $\Longrightarrow$ nowhere dense.

Other choices, other rooms?

	$\overline{\mathrm{d}}$	χ	ω
Minors			
Topological minors	Bounded expansion		Nowhere dense
Immersions			

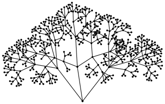
Definition

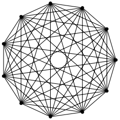
Other choices, other rooms?

	$\overline{\mathrm{d}}$	χ	ω
Minors	Bounded expansion	Bounded expansion	Nowhere dense
Topological minors	Bounded expansion	Bounded expansion	Nowhere dense
Immersions	Bounded expansion	Bounded expansion	Nowhere dense

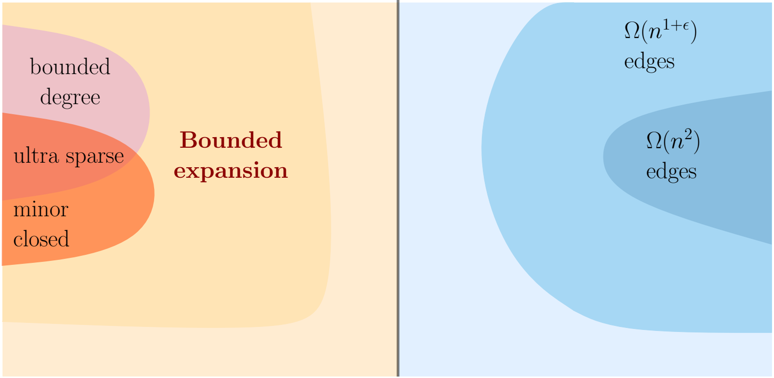
Theorem (Nešetřil, POM 2012)

Taxonomy of Classes



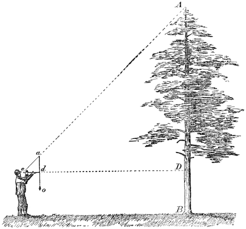


Nowhere dense | **Somewhere dense**



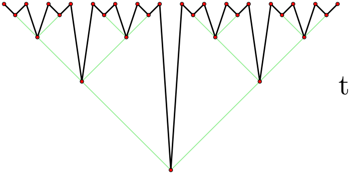
Tree-depth

Definition



The *tree-depth* td(G) of a graph G is the minimum height of a rooted forest Y s.t.

$$G \subseteq \text{Closure}(Y).$$



$$\text{td}(P_n) = \log_2(n+1)$$

Low tree-depth decompositions

$\chi_p(G)$ is the minimum of colors such that every subset I of $\leq p$ colors induces a subgraph G_I so that $\mathrm{td}(G_I) \leq |I|$.

Theorem (Nešetřil and POM; 2006, 2010)

$$\forall p,\ \sup_{G\in\mathcal{C}} \chi_p(G) < \infty \quad \iff \quad \mathcal{C} \text{ has bounded expansion.}$$

$$\forall p,\ \limsup_{G\in\mathcal{C}} \frac{\log \chi_p(G)}{\log |G|} = 0 \quad \iff \quad \mathcal{C} \text{ is nowhere dense.}$$

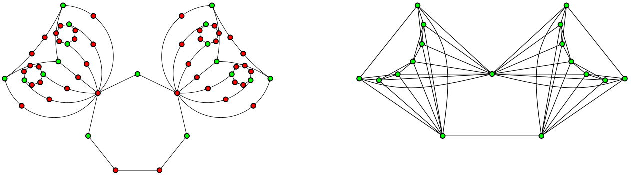
(extends DeVos, Ding, Oporowski, Sanders, Reed, Seymour, Vertigan on low tree-width decomposition of proper minor closed classes, 2004)

Distance Coloring

$G^{\sharp p}$: x and y adjacent if $\mathrm{dist}_G(x,y) = p$.

Theorem (Nešetřil, POM; 2006)

For every bounded expansion class $\mathcal{C}$ and every odd integer p it holds $\sup_{G\in\mathcal{C}} \chi(G^{\sharp p}) < \infty$.



Problem

How fast does $\sup_{G\in\mathcal{C}} \chi(G^{\sharp p})$ grow?

Restricted Homomorphism Dualities

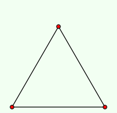
Theorem (Nešetril, POM; 2006)

Every class $\mathcal{C}$ with bounded expansion has all restricted dualities (ARD): $\forall F$ connected $\exists D$ such that $F \nrightarrow D$ and

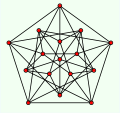
$$\forall G \in \mathcal{C}, \qquad (F \nrightarrow G) \iff (G \to D).$$

Example (Naserasr; 2007)

$\forall$ planar G



$\not\longrightarrow \quad G \qquad \Longleftrightarrow \qquad G \quad \longrightarrow$



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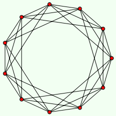
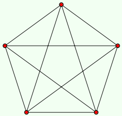
Example (Thomassen; 1994)

$\forall$ toroidal G



$\not\longrightarrow \quad G \quad \iff \quad G \quad \longrightarrow$



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Theorem (Nešetril, POM; 2012)

- For class $\mathcal{C}$ of graphs closed under subdivisions:
  $\mathcal{C}$ has ARD $\iff$ $\mathcal{C}$ has bounded expansion.
- For class $\mathcal{C}$ of directed graphs closed under reorientations:
  $\mathcal{C}$ has ARD $\iff$ $\mathcal{C}$ has bounded expansion.

Model Checking

Theorem (Dvořák, Kráľ, Thomas 2009; Grohe, Kreutzer 2011)

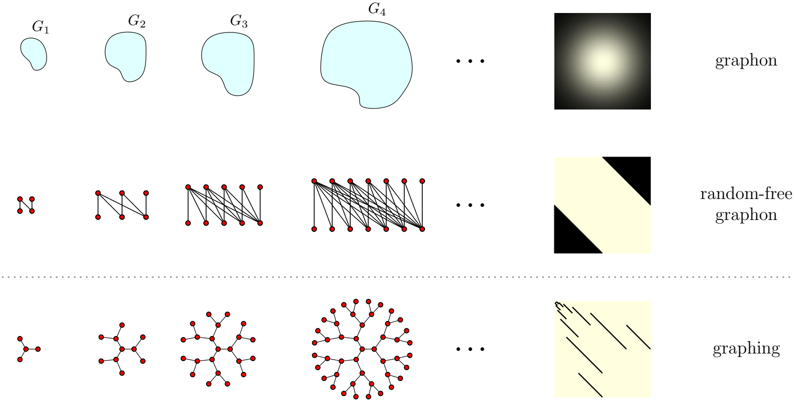
First-order properties may be checked in

- $O(n)$ time for G in a class with bounded expansion,
- $n^{1+o(1)}$ time for G in a class with locally bounded expansion.

Problem

Can first-order properties be checked in $O(n^c)$ time for G in a nowhere dense class?

Limits



Limits

Dense graphs

- *exchangeable random infinite graph*
- *graphon*

Aldous–Hoover–Kallenberg, Diaconis–Janson; Lovász–Szegedy

Bounded degree graphs

- *unimodular distribution*
- *graphing*

Benjamini–Schramm; Aldous–Lyons, Elek, Gaboriau

Structural Limits

Definition

A sequence $(G_n)_{n\in\mathbb{N}}$ is *FO-convergent* if, for every formula ϕ the probability $\langle\phi, G_n\rangle$ that G satisfies ϕ for a random assignment of the free variables converges.

Remark

This corresponds to convergence of measures μ_{G_n} associated to the G_n's on the Stone space of the Lindenbaum-Tarski algebra of FO.

A Limit Object for Sparse Classes?

A *modeling* $\mathbf{G}$ is a graph on a standard probability space s.t. every first-order definable set is measurable.

Theorem (Nešetřil, POM 2013+)

If a monotone class $\mathcal{C}$ has FO-limit modelings then $\mathcal{C}$ is nowhere dense.

Problem

Does every nowhere dense class has FO-limit modelings?

- true for bounded degree graphs (Nešetřil, POM 2012)
- true for bounded tree-depth graphs (Nešetřil, POM 2013)

Thank you for your attention.