ORDERINGS OF SPARSE GRAPHS

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PRAGUE

ERDÖS 100
BUDAPEST

ORDERINGS OF SPARSE GRAPHS

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(REMEMBERING DĚDEČEK)

ERDÖS 100
BUDAPEST

A WARM UP

EVERY ORIENTATION OF A GRAPH G
$\chi(G) \geq n+1$ CONTAINS A MONOTONNE
PATH $\vec{P}_{n+1}$ OF LENGTH n.

(GALLAI, HASSE, VITAVER, ROY)

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FOR EVERY ORIENTED TREE $\vec{T}$ THERE EXISTS ORIENTED GRAPH $\vec{H}$ SUCH THAT

$$\vec{T} \not\to G \quad \text{IFF} \quad G \longrightarrow \vec{H}$$

FOR EVERY ORIENTED GRAPH G.

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FOR EVERY ORIENTED GRAPH G.

(N., C. TARDIF)

$\boxed{\text{FINITE DUALITIES}}$ STUDIED BY MANY
(HUBIČKA, KANTOR, P. ERDŐS, J. FONIOK,
G. TARDOS, ...)

IN DUALITIES: $\boxed{\text{TREES ONLY}}$

IN RAMSEY CONTEX: ALL ORDERINGS
(~ ACYCLIC ORIENTATIONS)

$\boxed{\text{THM}}$ (ORDERING LEMMA)

FOR EVERY GRAPH $F = (V, E)$
THERE EXISTS GRAPH $G = (V', E')$
SUCH THAT:
FOR ARBITRARY ORDERINGS $(V, \leq)$
$(V', \leq')$
THERE EXISTS A $\boxed{\text{MONOTONNE}}$ (w.r.t. $\leq, \leq'$)
EMBEDDING $F \hookrightarrow G$.

("ALL GRAPHS HAVE ORDERING PROPERTY")

(N. RÖDL)

MORE THAN MEETS THE EYE

I. $\boxed{\text{THM}}$ HOLDS FOR GRAPHS WITH GIVEN GIRTH:

$\boxed{\text{THM}}$ (ORDERING LEMMA)

FOR EVERY GRAPH $F=(V,E)$, GIRTH$(F)>\ell$,
THERE EXISTS GRAPH $G=(V',E')$, GIRTH$(G)>\ell$,
SUCH THAT:

FOR ARBITRARY ORDERINGS $(V,\leq)$
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THERE EXISTS A $\boxed{\text{MONOTONNE}}$ (w.r.t. $\leq,\leq'$)
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(N, RÖDL, PROC.AMER.MATH.SOC.)

RANDOM REPLACEMENT CONSTR.

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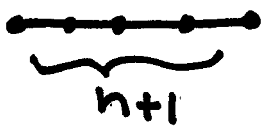
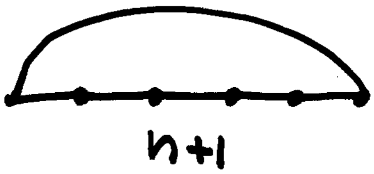
RANDOM REPLACEMENT CONSTR.

$(\text{"ALL } \underbrace{}_{\text{(SPARSE}}\text{ GRAPHS HAVE ORDERING PROPERTY"})$

COROLLARY

1. F = 

$\Rightarrow$ $\chi(G) \geq n+1$ (ERDŐS)

2. F = 

$\Rightarrow$ GIRTH(G) > n

G NOT DIAGRAM OF A POSET

(ERDŐS, ORE PROBLEM)

MORE COROLLARIES LATER

II. A CLASS $\mathcal{C}$ OF GRAPHS HAS $\boxed{\text{ORDERING PROPERTY}}$ IF $\boxed{\text{THM}}$ HOLDS FOR $\mathcal{C}$:

FOR EVERY $F \in \mathcal{C}$ THERE EXISTS $G \in \mathcal{C}$

$$G \xrightarrow[\text{ORD}]{} F$$

"ALMOST EVERY" CLASS $\mathcal{C}$ HAS ORDERING PROPERTY

(E.G. EVERY $\mathcal{C}$ DETERMINED BY FORBIDDING FINITELY MANY 2-CONNECTED GRAPHS)

CONSEQUENTLY ORDERINGS ARE NATURAL OBSTACLE FOR RAMSEY PROPERTIES.

(THE ORIGINAL MOTIVATION OF ORDERING PROPERTY)

$\boxed{\text{REMARK}}$

(SHELAH) ORDERING PROPERTY OF $\mathcal{C}$:

FOR EVERY n THERE EXISTS $G \in \mathcal{C}$
SUCH THAT IN G ONE CAN FO-DEFINE
A LINEAR ORDERING OF n-TUPLE

$\mathcal{C}$ IS $\boxed{\text{STABLE}}$ IF $\mathcal{C}$ HAS NO ORDERING PROPERTY

THM FOR A MONOTONNE CLASS $\mathcal{C}$

1. $\mathcal{C}$ IS STABLE
2. $\mathcal{C}$ IS NOWHERE DENSE

(N., P. OSSONA DE MENDEZ
ADLER, ADLER)

$\boxed{\text{THM}}$ FOR A MONOTONNE CLASS $\mathcal{C}$

1. $\mathcal{C}$ IS STABLE
2. $\mathcal{C}$ IS NOWHERE DENSE

(N., P. OSSONA DE MENDEZ)
ADLER, ADLER

NOWHERE DENSE $\equiv$ DOESN'T CONTAIN ALL K_n WITH $\leq k$ SUBDIVISION POINTS ON EVERY EDGE.

NOWHERE DENSE $\supseteq$ BOUNDED EXPANSION

$\boxed{\text{THM}}$

ANY BOUNDED EXPANSION CLASS $\mathcal{C}$
HAS $\boxed{\text{ALL RESTRICTED DUALITIES}}$:

FOR EVERY $F_1, \dots, F_t$ (CONNECTED) THERE EXIST D
SUCH THAT

1) $F_i \not\to D \quad i = 1, \dots, t$

2) FOR EVERY $G \in \mathcal{C}$

$$F_i \not\to G, \; i = 1, \dots, t \iff G \to D.$$

NOWHERE DENSE $\supseteq$ BOUNDED EXPANSION

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("ABSENCE OF ORDERINGS
LEADS TO DUALITIES")

Jaroslav Nešetřil • Patrice Ossona de Mendez

Sparsity

Graphs, Structures, and Algorithms

Springer

QUANTITATIVE VERSION

$[N] = \{1, 2, \ldots, N\}$

$k \ll N$

$\pi_1, \pi_2, \ldots, \pi_{k!}$ FIXED ENUMERATION OF ALL PERMUTATIONS OF $[k]$.

σ PERMUTATION OF $[N]$.

k - PROFILE
k - STATISTICS OF σ $\left(s_1^\sigma, \ldots, s_{k!}^\sigma\right)$

$$s_i = \left|\left\{K \in \binom{[N]}{k} ;\ \sigma|K \simeq \pi_i\right\}\right| \Big/ \binom{N}{k}$$

$\boxed{\text{THM}}$

FOR EVERY 2-CONNECTED F, $|F|=k$, THERE EXISTS G SUCH THAT

(1) FOR ANY σ ON $V(G)=[N]$:

$$\mathrm{Emb}\big((F,\pi_i),(G,\sigma)\big) = \left(s_i^{\sigma}+o(1)\right)\cdot \mathrm{Emb}(F,G)$$

(2) $\mathrm{GIRTH}(F)=\mathrm{GIRTH}(G)$.

"SPARSIFICATION LEMMA"

$\boxed{\text{THM}}$

FOR EVERY 2-CONNECTED F, $|F| = k$,
THERE EXISTS G, $V(G) = [N]$,
SUCH THAT FOR ANY σ ON $[N]$

$$\mathrm{Emb}((F, \pi_i), (G, \sigma)) = \left(1/k! + o(1)\right) \cdot \mathrm{Emb}(F, G)$$

AND GIRTH $(F) =$ GIRTH(G).

N. RÖDL

GENERALIZES O. ANGEL, A. KECHRIS, R. LYONS
(IN CONTEXT OF TOPOLOGICAL DYNAMICS)

FOR STRUCTURES WITH CANONICAL ORDERINGS ?

(N., PROMEL, RODL, VOIGT)

HOMOMORPHISMS OF ORDERED GRAPHS ?

⋮

THANK YOU
FOR YOUR ATTENTION