Super-expanders

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Expander sequences

A sequence of 3-regular graphs

$$\left\{G_n = \{\{1,\ldots,n\}, E_n\}\right\}_{n=1}^{\infty}$$

forms an expander sequence if

$$\inf_{n\geq 1} \min_{\substack{S\subset\{1,\ldots,n\}\\ 1\leq|S|\leq n/2}} \frac{\#\{\text{edges leaving } S\}}{|S|} > 0.$$

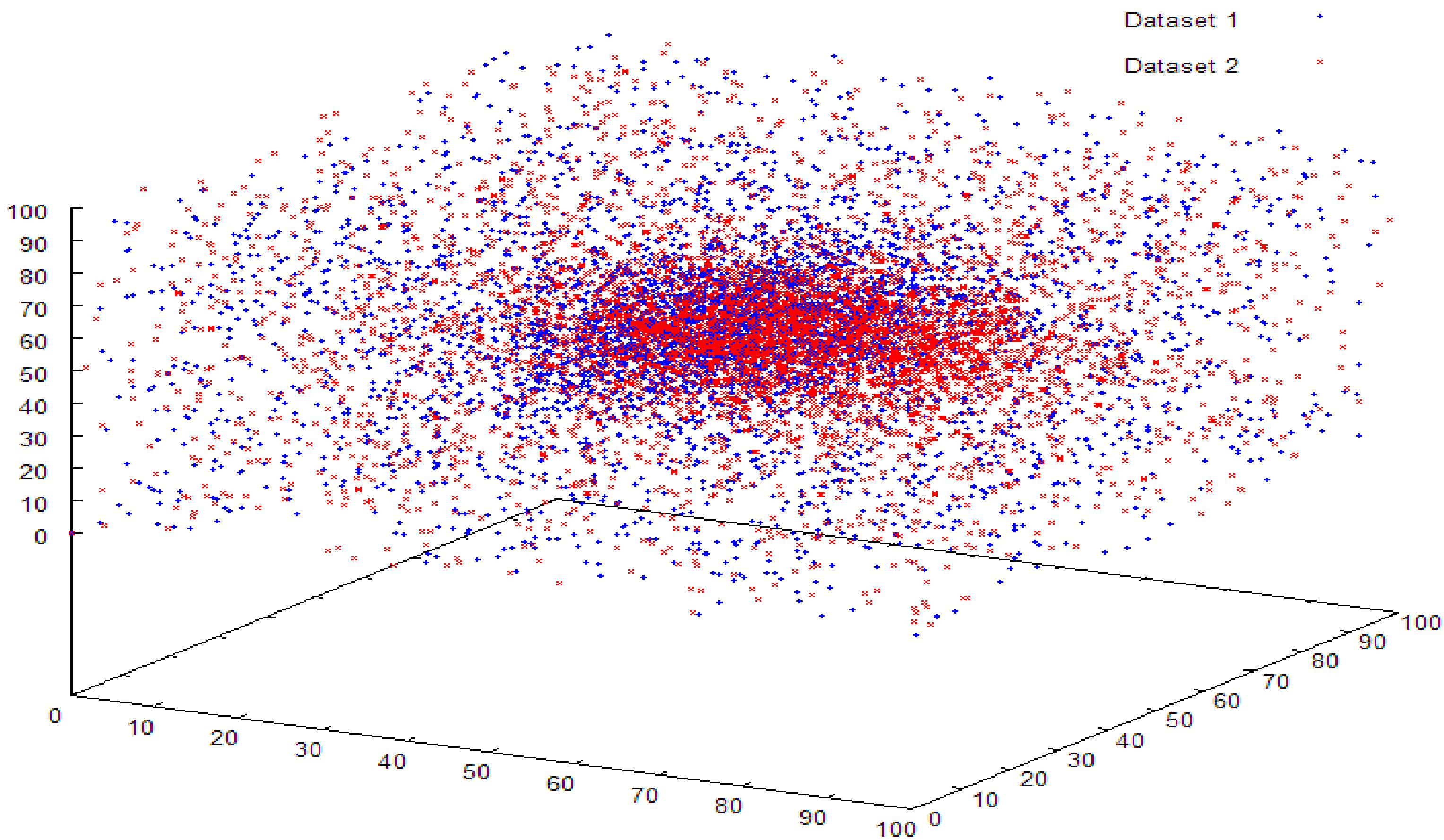
Geometric reformulation

There exists (equivalently for every) p>0 such that for every n vectors $v_1, \ldots, v_n \in \mathbb{R}^k$

$$\frac{1}{n^2}\sum_{i,j=1}^{n} \|v_i - v_j\|_2^p \asymp \frac{1}{n}\sum_{\{i,j\}\in E_n} \|v_i - v_j\|_2^p$$

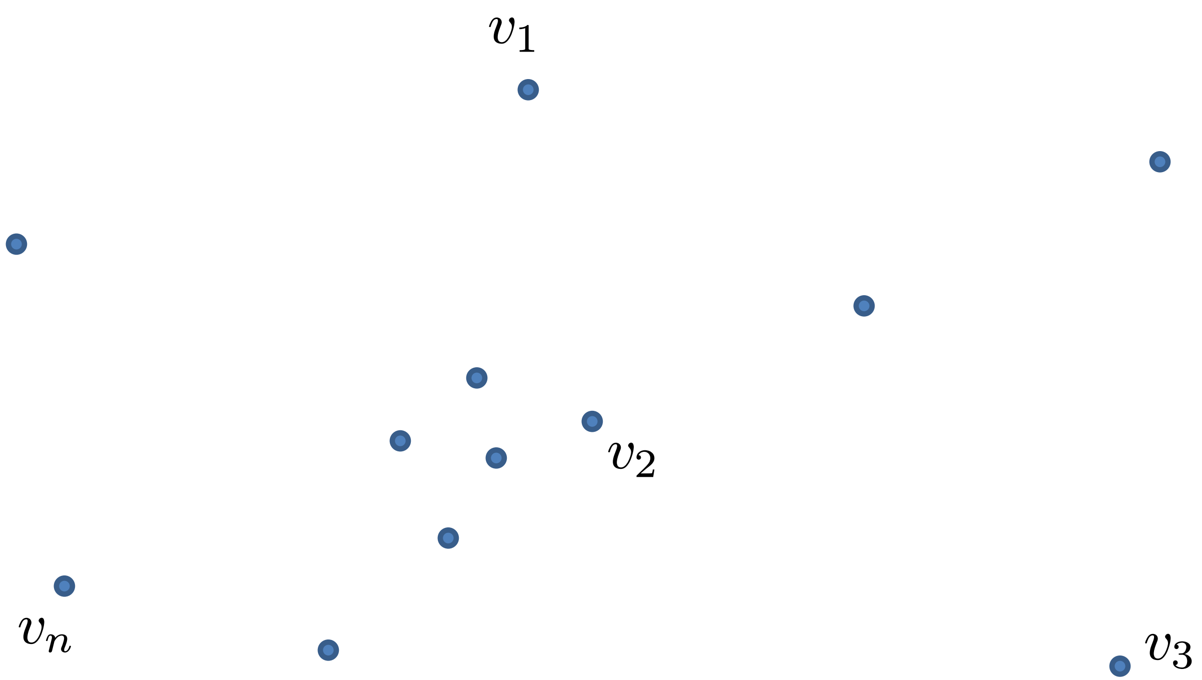
- p=1: the definition of expansion.
- p=2: spectral gap.
- Equivalence of p=1 and p=2: Cheeger's inequality.
- From now on in this talk: p=2.

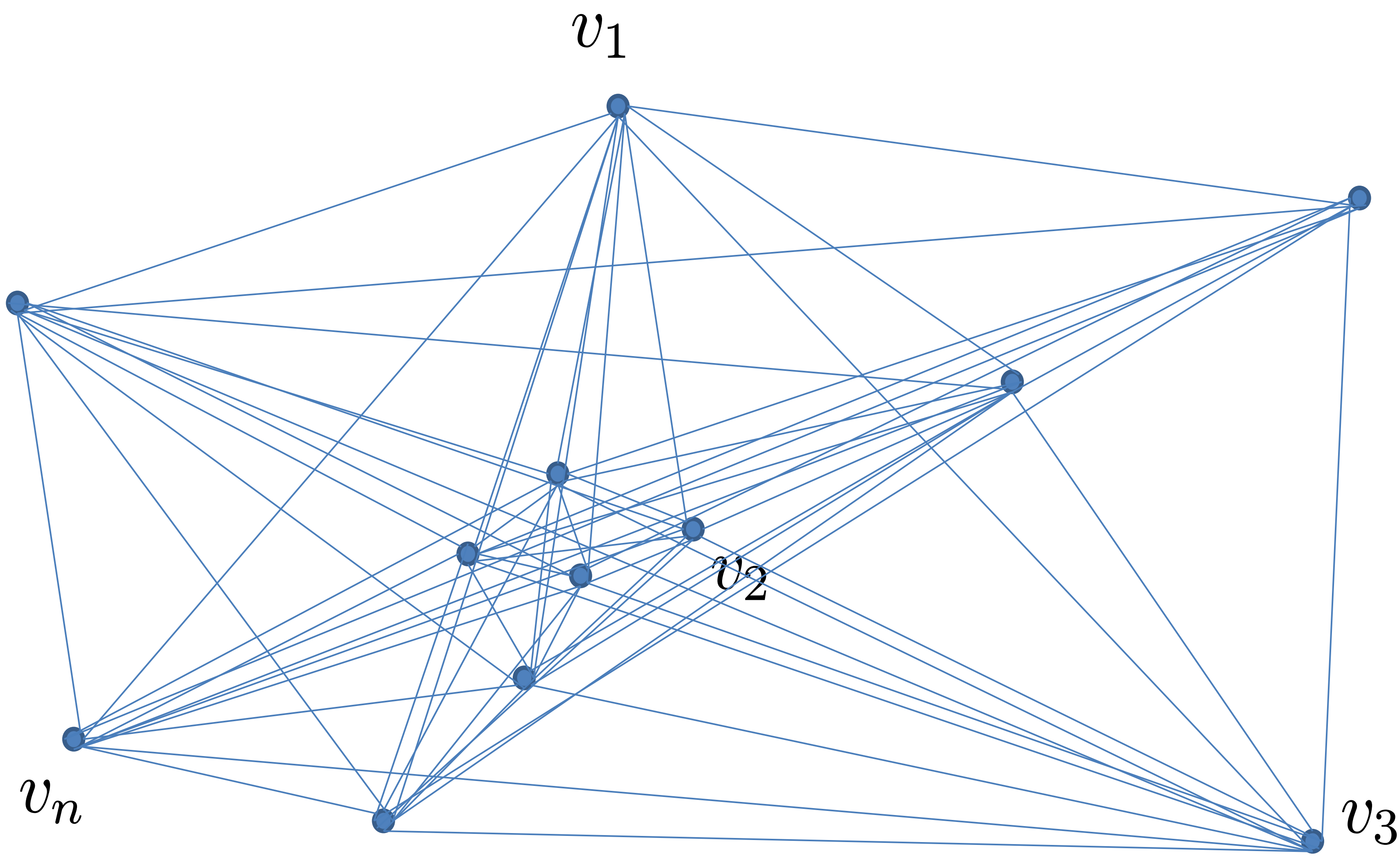
$$\frac{1}{n^2}\sum_{i,j=1}^{n}\|v_i - v_j\|_2^2 \asymp \frac{1}{n}\sum_{\{i,j\}\in E_n}\|v_i - v_j\|_2^2$$

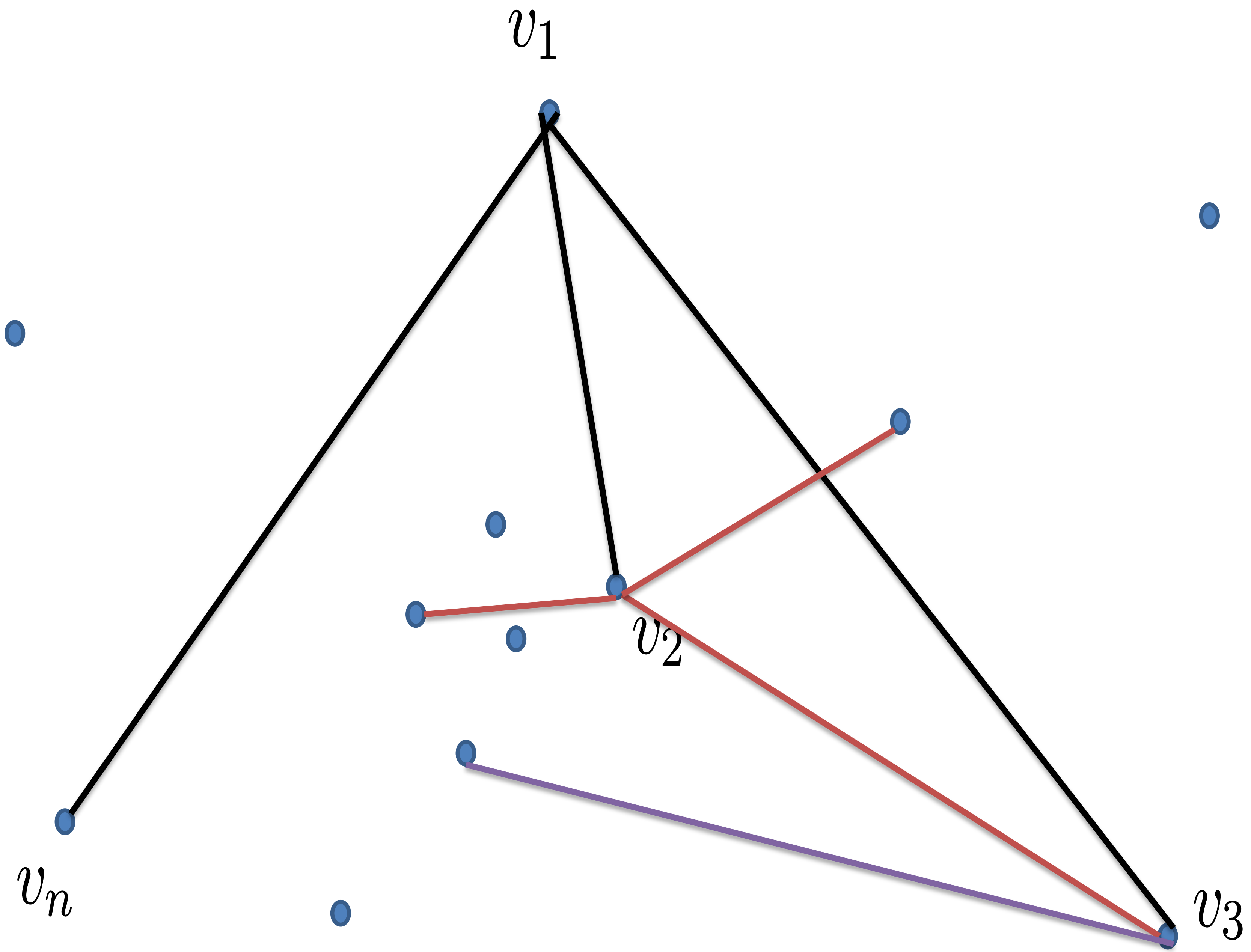


Universal approximators

Indyk (1999), Barhum-Goldreich-Shraibman (2007).




v_1
v_2
v_n
v_3


v_1
v_2
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v_3

Metric spaces other than $\mathbb{R}^k$?

A sequence of 3-regular graphs

$$\left\{G_n = \{\{1,\ldots,n\}, E_n\}\right\}_{n=1}^{\infty}$$

is an expander sequence with respect to a metric space (X,d) if for every $x_1,\ldots,x_n \in X$

$$\frac{1}{n^2}\sum_{i,j=1}^{n} d(x_i,x_j)^2 \asymp \frac{1}{n}\sum_{\{i,j\}\in E_n} d(x_i,x_j)^2$$

For every 3-regular graph $G = (\{1,\ldots,n\},E)$ and every metric space (X,d) we have

$$\frac{1}{n}\sum_{\{i,j\}\in E} d(x_i,x_j)^2 \lesssim \frac{1}{n^2}\sum_{i,j=1}^{n} d(x_i,x_j)^2$$

Nonlinear spectral gaps

Definition. Let $G=(\{1,\ldots,n\},E)$ be a graph and let (X,d) be a metric space. Then $\gamma(G,d^2)$ is the smallest $\gamma\in(0,\infty]$ such that for every $x_1,\ldots,x_n\in X$ we have

$$\frac{1}{n^2}\sum_{i,j=1}^{n} d(x_i,x_j)^2 \leqslant \frac{\gamma}{|E|}\sum_{\{i,j\}\in E} d(x_i,x_j)^2$$

Nonlinear spectral gaps

Definition'. Let $A=(a_{ij})$ be an n by n symmetric stochastic matrix and let (X,d) be a metric space. Then $\gamma(A,d^2)$ is the smallest $\gamma\in(0,\infty]$ such that for every $x_1,\ldots,x_n\in X$

$$\frac{1}{n^2}\sum_{i,j=1}^{n} d(x_i,x_j)^2 \leqslant \frac{\gamma}{n}\sum_{i,j=1}^{n} a_{ij}d(x_i,x_j)^2$$

- If A_G is the normalized adjacency matrix of G then $\gamma(A_G, d^2) = \gamma(G, d^2)$.

- If $1 = \lambda_1(A) \geq \lambda_2(A) \geq \ldots \lambda_n(A)$ are the eigenvalues of A then $$\gamma(A, d_{\mathbb{R}}^2) = \frac{1}{1 - \lambda_2(A)}$$

where $\forall\, x, y \in \mathbb{R}, \quad d_{\mathbb{R}} = |x - y|$.

- In general $\gamma(A, d^2)$ can be very different from $1/(1 - \lambda_2(A))$.

A sequence G_n of 3-regular graphs is an expander sequence with respect to (X,d) if

$$\sup_{n\geq 1} \gamma(G_n, d^2) < \infty$$

If X is not a singleton then this implies that G_n is an expander in the classical sense.

No constant-degree connected graph is an expander with respect to itself. ℓ_∞ does not admit an expander sequence.

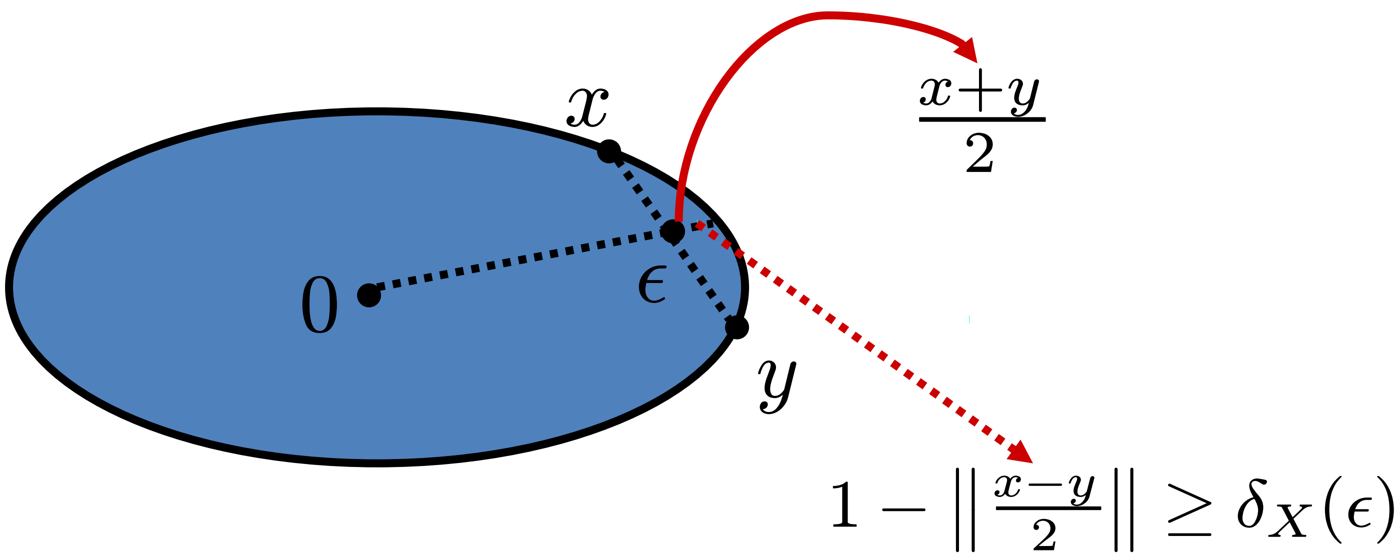
Nonlinear spectral gaps have been first defined formally by Gromov (2001), but they have been studied implicitly in embedding theory starting with Enflo (1976).

- Enflo (1976), Gromov (1983), Bourgain-Milman-Wolfson (1986), Pisier (1986), Linial-London-Rabinovich (1995), Matousek (1997).
- Gromov (2003), Ozawa (2004), Kasparov-Yu (2006), V. Lafforgue (2008,2009,2010), Pisier (2010), Mendel-N. (2010,2012,2013).
- Wang (1998,200), Gromov (2003), Izeki-Nayatani (2005), Pansu (2009), N.-Silberman (2011), Izeki-Kondo-Nayatani (2012), Liao (2013).
- Gromov (2001), Rabani-N. (2005), Pichot (2008).

Normed spaces: Kasparov-Yu problem

We saw that not all normed spaces admit expanders, but do uniformly convex normed spaces admit expanders?

(Uniform convexity for a normed space X:



)

Kasparov-Yu: does there exist a sequence of bounded degree graphs G_n that is an expander with respect to every uniformly convex normed space, i.e., for every uniformly convex normed space X,

$$\sup_{n\geq 1} \gamma(G_n, \|x-y\|^2) < \infty.$$

Such graphs are called super-expanders.

(their hope was that such graphs do not exist)

Lafforgue: Super-expanders do exist!

- Lafforgue (2008): Cayley graphs of finite quotients of co-compact lattices in $SL_3(\mathbb{Q}_p)$ are super-expanders.
- Liao (2013): Same for general connected almost simple algebraic groups with split rank at least 2 over non-Archimedian local field.
- Open: Is the same true for $SL_3(\mathbb{Z})$? (Margulis.)

Open: Are random regular graphs super-expanders with positive probability?

Open: Do there exist super-expanders whose girth is at least a constant multiple of their diameter?

Mendel-N. (2010): An inductive construction of super-expanders, using a modified zigzag iteration (Reingold-Vadhan-Wigderson, 2002).

A (significantly over-simplified) version of this iteration:

$$G_{n+1} = \left(\frac{1}{m} \sum_{t=0}^{m-1} G_n^t \right) ⓩ H.$$

One of the key tools: calculus for nonlinear spectral gaps:

$$\gamma\left(\frac{1}{m}\sum_{t=0}^{m-1} A^t, d^2\right) \lesssim \max\left\{1, \frac{\gamma(A,d^2)}{m}\right\}.$$

In Hilbert space this holds as equivalence:

$$\lambda\left(\frac{1}{m}\sum_{t=0}^{m-1} A^t\right) = \frac{1}{m}\sum_{t=0}^{m-1} \lambda(A^t)$$

$$= \frac{1}{m}\sum_{t=0}^{m-1} \lambda(A)^t = \frac{1-\lambda(A)^m}{m(1-\lambda(A))}$$

To prove such an inequality without using the algebraic (spectral) interpretation of $\gamma(A, d_{\mathbb{R}}^2)$, one needs to find a new proof even in Hilbert space.

Our proof relies on martingale methods, including Pisier's martingale cotype inequality for uniformly convex normed spaces in order to deduce K. Ball's metric Markov cotype property.

Barycentric metric spaces

Mendel-N. 2013: Nonlinear spectral calculus for metric spaces that are non-positively curved in the sense of Aleksandrov.

Every (finitely supported) probability measure μ on X admits a point $\mathfrak{B}(\mu) \in X$ such that for every point $x \in X$,

$$d(x, \mathfrak{B}(\mu))^2 \leqslant \int_X d(x,y)^2 d\mu(y) - c \int_X d(\mathfrak{B}(\mu), y)^2 d\mu(y)$$

Nonlinear martingales

Using a barycenter map one can define expectations and conditional expectations.

(S, μ) probability space and $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_n$ an increasing filtration of sigma algebras. Then $Z_0, \ldots, Z_n : S \to X$ is a martingale if

$$\mathfrak{B}(Z_i | \mathcal{F}_{i-1}) = Z_{i-1}.$$

Mendel-N. 2013: Pisier's martingale cotype inequality extends to this notion of martingale, and therefore non-positively curved spaces satisfy calculus for nonlinear spectral gaps.

$$\gamma\left(\frac{1}{m}\sum_{t=0}^{m-1}A^t, d^2\right) \lesssim \max\left\{1, \frac{\gamma(A,d^2)}{m}\right\}.$$

Question (N.-Silberman, 2004): Is true that for every non-positively curved metric space X, either every classical expander sequence is also an expander sequence with respect to X, or X does not admit any expander sequence?

Theorem (Mendel-N., 2013). There exists a non-positively curved metric space (X,d) that admits a sequence of 3-regular expanders G_n.

Yet, if H is a uniformly random n-vertex d-regular graph then

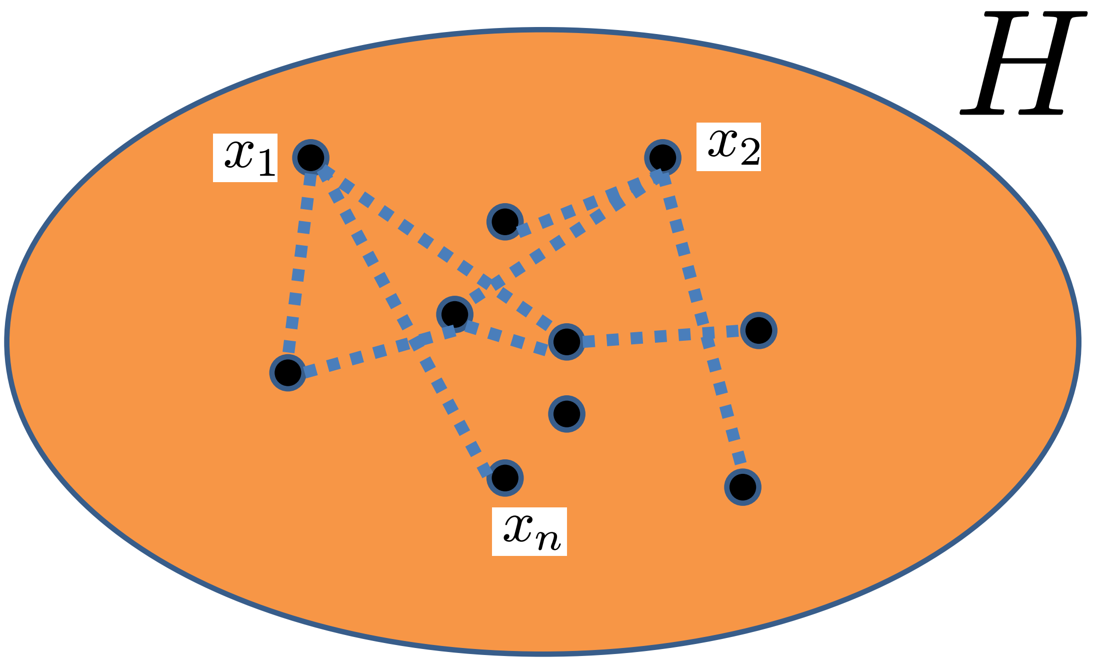
$$\Pr\left[\gamma(H,d^2) \geq c(\log_d n)^2\right] \geq 1 - \frac{C(d)}{\sqrt[3]{n}}.$$

Expanders for random graphs

Theorem (Mendel-N., 2013). The graphs G_n of the previous theorem have the property that if H is a uniformly random m-vertex d-regular graph and d_H denotes the shortest-path metric on H then

$$\Pr\left[\sup_{n\geq 1}\gamma(G_n, d_H^2)\leqslant K\right]\geq 1-\frac{C(d)}{\sqrt[3]{m}}.$$

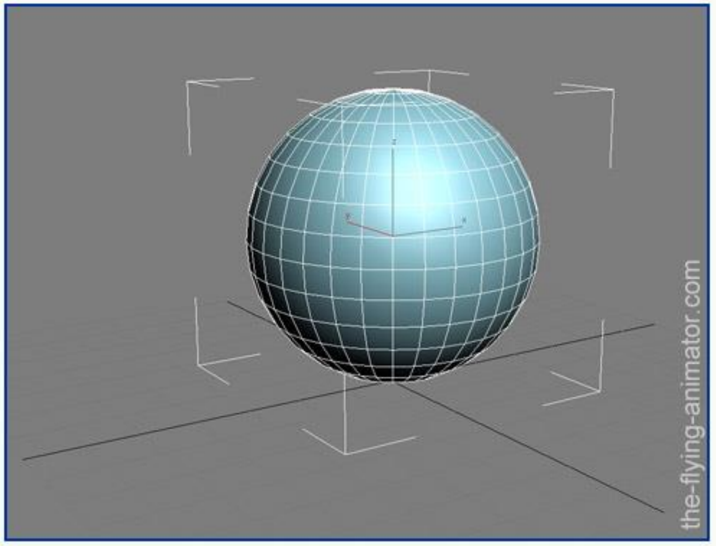
Universal approximator for random graphs



$$\frac{1}{n^2}\sum_{i,j=1}^{n} d_H(x_i,x_j)^2 \asymp \frac{1}{n}\sum_{\{i,j\}\in E_H} d_H(x_i,x_j)^2$$

The Euclidean cone

$$\mathbb{R}^n \backslash \{0\} = (0, \infty) \times S^{n-1}$$



$$\|s \cdot x - t \cdot y\|_2$$

$$= \sqrt{s^2 + t^2 - 2st\cos\left(d_{S^{n-1}}(x, y)\right)}.$$

Berestovskii (1983): The Euclidean cone over a metric space (X,d) is the completion of $(0,\infty)\times X$ under the metric

$$d_{\mathrm{Cone}(X)}\big((s,x),(t,y)\big)$$
$$=\sqrt{s^2+t^2-2st\cos\left(\min\{\pi,d(x,y)\}\right)}.$$

It turns out that the Euclidean cone over a random graph is close enough to a non-positively curved metric space to allow for the spectral calculus strategy to succeed.

A structure theorem for the Euclidean cone over a random graph

Theorem (Mendel-N., 2013). There exists a non-positively curved metric space X such that with probability at least $1-C(d)/\sqrt[3]{n}$ if H is a uniformly random n-vertex d-regular graph then one can write $H=A_1\cup A_2$ and there is $\sigma>0$ such that

$$\mathrm{Cone}(A_1,\sigma\cdot d_H)\hookrightarrow L_1$$

$$\mathrm{Cone}(A_2,\sigma\cdot d_H)\hookrightarrow X$$