Intersection theorems for finite sets

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Extremal Set Theory

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What if M misses only one number?

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Suppose that $\mathcal{A} \subset 2^{[n]}$ *and* $|A \cap B| \neq n/4$ *for all* $A, B, \in \mathcal{A}$, *and* $n > n_0$. *Then*

$$|\mathcal{A}| < (1.99)^n.$$

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Frankl-Rödl show it is about $(t/n)^2/2$.

Applications

- Combinatorics (solved Erdős-Szemerédi weak delta system conjecture)
- Geometry (solved Larman-Rogers conjecture, Borsuk problem)
- Coding Theory (improved Frankl-Blokhuis bound)
- Communication Complexity (Sgall 1999)
- Quantum Computing (Buhrman-Cleve-Wigderson 1998)
- Semidefinite Programming (Goemans-Kleinberg 1998, Hatami-Magen-Markakis 2009)

Katona's Theorem

Suppose we forbid all numbers less than $t+1$ as intersection sizes.

Define $\mathcal{A}(n,t)$ to be

$$\{A \subset [n] : |A| \geq (n+t+1)/2\} \quad \text{if } n+t \text{ is odd}$$

$$\{A \subset [n] : |A \cap ([n]-\{1\})| \geq (n+t)/2\} \text{ if } n+t \text{ is even.}$$

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Theorem (Katona)

Let $\mathcal{A} \subset 2^{[n]}$ and suppose that $|A \cap A'| > t$ for every $A, A' \in \mathcal{A}$. Then

$$|\mathcal{A}| \leq |\mathcal{A}(n,t)|.$$

Moreover, if $t \geq 1$ and $|\mathcal{A}| = |\mathcal{A}(n,t)|$, then $\mathcal{A} = \mathcal{A}(n,t)$.

The optimal ε_0

Conjecture

Let $0 < \eta < 1/3$, $\eta n < t < n/3$, and $\mathcal{A} \subset 2^{[n]}$ with $|A \cap B| \neq t$ for all $A, B \in \mathcal{A}$. Then

$$|\mathcal{A}| \leq \binom{n}{(n+t)/2} 2^{o(n)}.$$

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For $n/3 < t < (1/2 - \eta)n$, the construction $\mathcal{A} = \binom{[n]}{t}$ is better, and we conjecture it is optimal.

Forbidding a small interval

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Theorem (M-Rödl)

Let $0 < \varepsilon < 1/5$ *be fixed,* $n > n_0(\varepsilon)$, $\varepsilon n < t < n/5$ *and* $\mathcal{A} \subset 2^{[n]}$. *Suppose that* $|A \cap B| \notin (t, t + n^{0.525})$ *for all* $A, B \in \mathcal{A}$. *Then*

$$|\mathcal{A}| < n\binom{n}{(n+t)/2}.$$

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- The constant 0.525 is a consequence of the result of Baker-Harman-Pintz that there is a prime in every interval $(s - s^{0.525}, s)$ as long as s is sufficiently large.
- If we assume the Riemann Hypothesis, then 0.525 could be improved to $1/2 + o(1)$ using a result of Cramér.

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Suppose that $\mathcal{A} \subset 2^{[n]}$ is M-intersecting, where $M = \{0, 2, 4, \ldots\}$. In other words, $|A \cap B|$ is even for all $A, B \in \mathcal{A}$. Then $|\mathcal{A}| \leq 2^{\lfloor n/2 \rfloor} + 1$.

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Theorem (Eventown Theorem)

Suppose that $\mathcal{A} \subset 2^{[n]}$ such that

- *$|A|$ is even for every $A \in \mathcal{A}$*
- *$|A \cap B|$ is even for every $A, B \in \mathcal{A}$*

Then $|\mathcal{A}| \leq 2^{\lfloor n/2 \rfloor}$.

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The length $\ell(M)$ of a set M is the maximum number of consecutive integers contained in M.

$\ell(M) \leq \ell$ if and only if $\overline{M}$ is $(\ell + 1)$-syndetic.

Bounds for small $\ell(M)$

Theorem (M-Rödl)

Let $M \subset \{0, 1, \ldots, n\}$ with $\ell(M) = \ell$. Suppose that $\mathcal{A} \subset 2^{[n]}$ is an M-intersecting family. Then

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- The 1.622 is probably not sharp, just a result of the proof

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Let $M \subset \{0, 1 \ldots, n\}$ with $\ell(M) = \ell$. Suppose that $\mathcal{A} \subset 2^{[n]}$ is an M-intersecting family. Then

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- This is the first non-linear-algebraic proof of an asymptotic version of the Eventown Theorem; it applies in more general scenarios though doesn't give bounds as precise as $2^{n/2}$.

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Theorem (M-Rödl)

Let $M \subset \{0, 1 \ldots, n\}$ with $\ell(M) = \ell$. Suppose that $(\mathcal{A}, \mathcal{B})$ is an M-intersecting pair of families in $2^{[n]}$. Then

$$|\mathcal{A}||\mathcal{B}| < \min\left\{2.631^n \times 10^{4\ell}, \quad 2^{n+2\ell \log^2 n}\right\}.$$

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(A4) if $h(L), h(L') \leq s < \infty$, then either

$$h(L' \cap L) \leq s - 1 \qquad \text{or} \qquad h(L' \cap (L - 1)) \leq s - 1.$$

Sgall's theorem

Theorem (Sgall (1999))

Suppose that $(\mathcal{A}, \mathcal{B})$ is an M-intersecting pair of families in $2^{[n]}$ and $h(M) \leq s \leq n+1$. Then

$$|\mathcal{A}||\mathcal{B}| \leq 2^{n+s-1}\binom{n}{s-1}.$$

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Theorem (M-Rödl)

There exists a height function h *such that for* $M \subset \{0, 1 \ldots, n\}$,

$$h(M) \leq 1 + 2\ell(M) \log n.$$

Applying this bound in Sgall's Theorem yields $|\mathcal{A}||\mathcal{B}| < 2^{n+2\ell \log^2 n}$.

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- $h(L) = 1 + \max\{a, b\}$

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Thank You