Singular integrals on self-similar subsets of metric groups

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Hilbert transform

- $Hf(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \frac{f(y)}{y-x} dy$
- $H^* f(x) = \sup_{\epsilon > 0} | \int_{|x-y|>\epsilon} \frac{f(y)}{y-x} dy |$
- $H^* : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is bounded

Cauchy transform

- μ a finite Borel measure on $\mathbb{C}$
- $C_\mu^* f(z) = \sup_{\epsilon>0} |\int_{|z-w|>\epsilon} \frac{f(w)}{w-z} d\mu w|$
- When is $C_\mu^* : L^2(\mu) \to L^2(\mu)$ bounded?
- $\int C_\mu^*(f)^2 d\mu \lesssim \int |f|^2 d\mu$?

Regular sets and measures

μ is (Ahlfors-David) m-regular if

$$r^m/C \leq \mu(B(x,r)) \leq Cr^m \text{ for all } x \in \operatorname{spt}\mu, 0 < r < \operatorname{diam}(\operatorname{spt}\mu).$$

E is m-regular if

$$r^m/C \leq \mathcal{H}^m(E \cap B(x,r)) \leq Cr^m \text{ for all } x \in E, 0 < r < \operatorname{diam}(E).$$

Cauchy transform

Theorem (Mattila, Melnikov, Verdera, 1996, David,...)

Suppose μ is a 1-regular finite Borel measure on $\mathbb{C}$. Then $C_\mu^* : L^2(\mu) \to L^2(\mu)$ is bounded if and only sptμ is contained in a 1- regular curve.

Examples: $C_\mu^* : L^2(\mathcal{H}^1 \lfloor E) \to L^2(\mathcal{H}^1 \lfloor E)$ is unbounded for all fractal self-similar 1-dimensional sets E with open set condition (thanks for Andras Mathe for a correction here)

Removable sets

Theorem (Mattila, Melnikov, Verdera, 1996)

Suppose $E \subset \mathbb{C}$ *is a compact* 1*-regular set. Then the following are equivalent:*

E *is removable for bounded analytic functions*

E *is removable for Lipschitz harmonic functions*

$\mathcal{H}^1(E \cap \Gamma) = 0$ *for every rectifiable curve* Γ

Removable sets

Examples: all fractal self-similar 1-dimensional sets with open set condition

Much more general results were later proven by David, Nazarov, Treil, Volberg, Tolsa, and others

Riesz transforms

- $0 < m < n, \mu$ m-regular
- $R^*_{m,\mu} f(x) = \sup_{\epsilon>0} |\int_{|x-y|>\epsilon} \frac{y-x}{|y-x|^{m+1}} f(y) d\mu y|, x \in \mathbb{R}^n$
- David and Semmes, 1991: $R^*_{m,\mu} : L^2(\mu) \to L^2(\mu)$ is bounded for uniformly rectifiable m-regular measures μ
- Conjecture: converse holds
- Vihtilä, 1996: $R^*_{m,\mu} : L^2(\mu) \to L^2(\mu)$ is not bounded if m is not an integer

Riesz transforms

David-Semmes conjecture is true when $m = n - 1$:

Theorem (Nazarov, Tolsa, Volberg, 2013)

Suppose μ is an $(n-1)$-regular finite Borel measure on $\mathbb{R}^n$. Then $R^*_{m,\mu} : L^2(\mu) \to L^2(\mu)$ is bounded if and only sptμ is an $(n-1)$-dimensional uniformly rectifiable set.

Riesz transforms

Theorem (Nazarov, Tolsa, Volberg, 2013)

Suppose $E \subset \mathbb{R}^n$ *is a compact* $(n-1)$*-regular set. Then* E *is removable for Lipschitz harmonic functions, if and only* $\mathcal{H}^{n-1}(E \cap S) = 0$ *for every* $(n-1)$*-dimensional* C^1 *surface* S*.*

General setting

- X a separable metric space, μ a finite Borel measure on X,
- $K : X \times X \setminus \{(x,y) : x = y\} \to \mathbb{R}$ Borel function,
- $K(x,y) = -K(y,x)$,
- K bounded in $\{(x,y) : |x-y| > \delta\}$ for every $\delta > 0$,
- $T^*_{K,\mu}f(x) = \sup_{\epsilon>0} |\int_{d(x,y)>\epsilon} K(x,y)f(y)d\mu y|, x \in X$.

Convergence

When does L^2-boundedness;

$$\int T^*_{K,\mu}(f)^2 d\mu \lesssim \int f^2 d\mu,$$

imply almost everywhere convergence of principal values;

$$\exists \lim_{\epsilon \to 0} \int_{X \setminus B(x,\epsilon)} f(y)K(x,y)d\mu y \text{ for } \mu \text{ almost all } x?$$

Convergence

- When $X = \mathbb{R}^n$ and K is the Riesz kernel, results of Mattila and Preiss, 1994, and Tolsa, 2008, say that the almost everywhere convergence of principal values implies rectifiability.
- Similar results by Huovinen, 1997, for other kernels, for example $z^{2k-1}/|z|^{2k}, z \in \mathbb{C}, k \in \mathbb{N}$.

L^2-boundedness implies rectifiability with the kernels $x^{2k-1}/|z|^{2k}, z = x + iy \in \mathbb{C}$, by Chousionis, Mateu, Prat and Tolsa, 2012.

Convergence

Theorem (Mattila and Verdera, 2009)

If $\int T^*_{K,\mu}(f)^2 d\mu \lesssim \int f^2 d\mu$*, then the operators*

$$T_{K,\mu,\epsilon}f(x) = \int_{X\setminus B(x,\epsilon)} K(x,y)f(y)d\mu y,$$

converge weakly in $L^2(\mu)$*;* $T_{K,\mu,\epsilon} \to T_{K,\mu}$ *as* $\epsilon \to 0$*, and*

$$T_{K,\mu}f(x) = \lim_{\epsilon\to 0} \frac{1}{\mu B(z,r)} \int_{B(z,r)} \int_{X\setminus B(z,r)} K(x,y)f(y)d\mu y d\mu x$$

for μ *almost all* $z \in X$.

Unfortunately we don't know for the Riesz kernels if the above converge implies rectifiability nor if the L^2-boundedness implies the almost everywhere convergence of principal values, except when $m = 1$ or $m = n - 1$.

Metric groups

- (G, d) is a complete separable metric group with the following properties:
- The left translations $\tau_q : G \to G$, $\tau_q(x) = q \cdot x, x \in G$, are isometries for all $q \in G$.
- There exist dilations $\delta_r : G \to G, r > 0$, which are continuous group homomorphisms for which,
- $\delta_1 =$ identity, $\delta_{rs} = \delta_r \circ \delta_s$,
- $d(\delta_r(x), \delta_r(y)) = rd(x, y)$ for $x, y \in G, r > 0$,

Self-similar sets

Let $\mathcal{S} = \{S_1, \dots, S_N\}, N \geq 2$, be an iterated function system of similarities of the form $S_i = \tau_{q_i} \circ \delta_{r_i}$ where $q_i \in G, r_i \in (0,1)$ and $i = 1, \dots, N$. The self-similar set C is the invariant set with respect to $\mathcal{S}$, that is, the unique non-empty compact set such that

$$C = \bigcup_{i=1}^{N} S_i(C).$$

Self-similar sets

Suppose that the sets $S_i(C)$ are pairwise disjoint for $i = 1, \dots, N$. Then

$$0 < \mathcal{H}^s(C) < \infty \text{ for } s \text{ such that } \sum_{i=1}^{N} r_i^s = 1,$$

and the measure $\mu = \mathcal{H}^s \lfloor C$ is s-regular.

Self-similar sets

Let $K : G \setminus \{e\} \to \mathbb{R}$ be an s-homogeneous kernel, $K(\delta_r(x)) = r^{-s}K(x)$ and

$$T^*_{K,\mu}f(x) = \sup_{\epsilon>0} \left| \int_{d(x,y)>\epsilon} K(x^{-1}y)f(y)d\mu y \right|, x \in G.$$

Self-similar sets

Theorem (Chousionis and Mattila, 2012)

If there exists a fixed point x_w *for some* $S_w = S_{w_1} \circ \cdots \circ S_{w_k}, w = (w_1, \ldots, w_k)$, *such that*

$$\int_{C \setminus S_w(C)} K(x_w^{-1} y) d\mathcal{H}^s y \neq 0,$$

then the maximal operator $T^*_{K, \mathcal{H}^s \lfloor C}$ *is unbounded in* $L^2(\mathcal{H}^s \lfloor C)$, *moreover* $\| T^*_K(1) \|_{L^\infty(\mathcal{H}^s \lfloor C)} = \infty$.

Self-similar sets

Is it true in the above situation that L^2-boundedness implies the almost everywhere convergence of principal values?

This question is (vaguely) of the following type: when does the boundedness of the ergodic sums

$$\sum_{j=1}^{m} f(\sigma^j x)$$

imply their almost everywhere convergence?

Self-similar sets

Example where this fails:
$\sigma(\omega_1, \omega_2, \dots) = (\omega_2, \omega_3, \dots), \omega_j \in \{0, 1\}$, the shift,
μ the standard uniform measure on $\{(\omega_j) : \omega_j \in \{0, 1\}\}$,
$f(\omega_j) = 1$, if $\omega_1 = 0, \omega_2 = 1$, $f(\omega_j) = -1$, if $\omega_1 = 1, \omega_2 = 0$,
and $f(\omega_j) = 0$ otherwise.

Heisenberg group $\mathbb{H}$

Heisenberg group $\mathbb{H}$ is $\mathbb{R}^3$ equipped with a non-abelian group structure, with a left invariant metric and with natural dilations.

Heisenberg group $\mathbb{H}$

- $\mathbb{H} = \mathbb{C} \times \mathbb{R}$, $p = (w, s), q = (z, t) \in \mathbb{H}$
- $p \cdot q = (w + z, s + t + 2Im(w\bar{z}))$
- $||p|| = (|z|^4 + t^2)^{1/4}$
- $d(p, q) = ||p^{-1} \cdot q|| = (|w - z|^4 + |s - t + 2Im(w\bar{z})|^2)^{1/4}$
- $\delta_r(p) = (rz, r^2 t)$
- $d(\delta_r(p), \delta_r(q)) = rd(p, q)$
- $d(p \cdot q_1, p \cdot q_2) = d(q_1, q_2)$
- $\dim_H \mathbb{H} = 4$

Horizontal differential operators

- $p = (z, t) = (x + iy, t) \in \mathbb{H}$
- $X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}$, $Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}$
- $\nabla_H = (X, Y)$
- $\Delta_H = X^2 + Y^2$
- f is Δ_H-harmonic if $\Delta_H f = 0$.

Fundamental solution and the kernel K

- The fundamental solution for $\Delta_H f = 0$ is $\Gamma(p) = c||p||^{-2}$.
- $\nabla_H \Gamma(p) = c(\frac{x(x^2+y^2)+yt}{||p||^6}, \frac{y(x^2+y^2)-xt}{||p||^6}), p = (x+iy, t)$
- $K(p) = \nabla_H \Gamma(p)$
- $T^*_{K,\mu} f(p) = \sup_{\epsilon>0} || \int_{d(p,q)>\epsilon} K(p^{-1}q) f(q) d\mu q||, p \in \mathbb{H}$
- $T^*_{K,\mu}$ is unbounded on $L^2(\mu)$ for many self-similar measures μ

Removable sets

- A compact set $E \subset \mathbb{H}$ is removable for Lipschitz Δ_H-harmonic functions if for all open sets U with $E \subset U$ every Lipschitz function $f : U \to \mathbb{R}$ which is Δ_H-harmonic in $U \setminus E$ is harmonic in U.
- If $\mathcal{H}^3(E) = 0$, then E is removable.
- If $\dim E > 3$, then E is not removable.
- Many self-similar 3-dimensional sets are removable for Lipschitz Δ_H-harmonic functions.