Smooth numbers and zeros of Dirichlet L- functions

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1 Introduction

We investigate relations between sums of the form

$$\sum_{M<p\leq M'} \chi(p)p^{+it},$$

where χ is a non- principal character mod$Q$, $t \in \mathbb{R}$, $M < M' \leq 2M$ and zero- free regions of the related Dirichlet L- function

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s}.$$

One direction can easily be obtained by complex integration as a generalization of the Explicit formula

$$\psi(x,\chi) = E_0(\chi)x - \sum_{|\Im(\rho)|\leq T} \frac{x^\rho}{\rho} + O_Q\left(\frac{x\log^2 x}{T}\right),$$

where $\psi(x,\chi) = \sum_{n\leq x} \Lambda(n)\chi(n)$ with von- Mangoldt's function Λ and

$$E_0(\chi) = \begin{cases} 1, & \chi = \chi_0 \\ 0, & \chi \neq \chi_0. \end{cases}$$

We set

$$\psi(x,\chi,t) = \sum_{n\leq x} \Lambda(n)\chi(n)n^{+it_0}$$

and obtain

$$\psi(x,\chi,t_0) = E_0(\chi)x^{1-it_0} - \sum_{|\Im(\rho)-t_0|\leq T} \frac{x^{\rho-it_0}}{\rho - it_0} + O\left(\frac{x\log^2(xt_0)}{T}\right). \tag{1}$$

The result for $\sum_{p\leq x} \chi(p)p^{it_0}$ follows by partial summation.
We obtain Theorem 1:
Assume that $L(s,\chi) \neq 0$ for $\Re(\rho) \geq \sigma_0$. Then

$$\sum_{M<p\leq 2M} \chi(p)p^{it_0} \ll_Q M^{\sigma_0}\log^2|t_0|.$$

Results in the other direction were obtained by Turán (1974, see []) by the application of his power- sum method.
We just cite one example:
Theorem (Turán):
Suppose the existence of constants $\alpha \geq 2$, $0 < \beta \leq 1$ and $c(\alpha,\beta)$ so that $\tau > c(\alpha,\beta)$ the inequality

$$\left|\sum_{N_1\leq p\leq N_2} \exp(-i\tau\log p)\right| \leq \frac{N\log^{10} N}{\tau^\beta}$$

holds for all N_1, N_2 integers with $\tau^\alpha \leq N \leq N_1 < N_2 \leq 2N \leq \exp(\tau^{\beta/10})$. Then $\zeta(s) \neq 0$ on the segment

$$\sigma > 1 - \frac{e^{-10}\beta^3}{\alpha^2}$$

with $t = \tau$ and $s = \sigma + it$.

We shall prove

Theorem 2:

Let $Q \in \mathbb{N}$ and $Q > 1$. Let $B = B(Q) > 0$ a fixed but arbitrarily large constant. Let χ be a non- principal Dirichlet- character mod$Q$. Let $\ell = log(|t| + A) + B$ for $t \in \mathbb{R}$ and $A \geq \frac{1}{2}$. Assume that the following hypothesis holds:

$$\sum_{M<p\leq M'} \chi(p)p^{it} \leq M^{\sigma_0}$$

for $M < M' < 2M$, $M \geq \ell^A$ and $\sigma_0 = 1 - \frac{1}{A}$.

Then $L(s,\chi) \neq 0$ for $\sigma > \sigma_0 + \epsilon$, where $\epsilon = \epsilon(B) \to 0$ for $B \to 0$.

2 Proof of Theorem 1

A standard application of Perron's formula gives (1).

Let

$$N(T,\chi) = \{\rho \colon L(s,\chi) = 0,\ 0 \leq \Im(\rho) \leq T\}.$$

By the well- known estimate $N(T+1,\chi) - N(T,\chi) = O_Q(\log T)$, (see []) we obtain from (1) with $x = M$, $x = M'$, $T = t_0$

$$\sum_{M<n\leq M'} \Lambda(n)\chi(n)n^{it_0} \ll_Q M^{\sigma_0}\log^3|t_0|.$$

Theorem 1 follows by partial summation.

3 Approximation by Dirichlet- polynomials

Definition 3.1:

$$L_x(s,\chi) = \sum_{1\leq n\leq x} \chi(n)n^{-s}.$$

Lemma 3.1:

Let $C_1 > 1$, $\sigma > 0$, $s = \sigma + it$ and $|t| \leq \frac{2\pi x}{C_1}$. Then

$$L(s,\chi) = L_x(s,\chi) + O_{\sigma,C_1}(x^{-\sigma}).$$

Proof: Karacuba, [].

4 Construction of the Mollifier

We start with a partition of the set of integers into boxes, cartesian products of intervals for the prime factors of these integers.

Definition 4.1 (The boxes):

Let $\mathcal{L} = \ell^A$ and ℓ^ℓ.

We partition the interval $[\mathcal{L}, x]$ into subintervals I_j. For this purpose we define the sequence (Y_j) by

$$y_j = \mathcal{L} \cdot 2^j, \tag{2}$$

with $j \in \mathbb{N}_0$ and $0 \leq j \leq J_0$, where $J_0 = \{\min j \colon \mathcal{L} \cdot 2^j \geq x\}$ and set $I_j = [y_j, y_{j+1}]$.

Let $\nu(m, I_j)$ be the number of primefactors of m ($\mu^2(m) = 1$) in the interval I_j. Let m_0 be an integer consisting only of primes $p \leq \mathcal{L}$:

$$p | m_0 \Rightarrow p \leq \mathcal{L}.$$

For each $n \in \mathbb{N}$ with $\mu^2(n) = 1$, we set

$$m_0(n) = \prod_{\substack{p|n \\ p \leq \mathcal{L}}} p.$$

Let $\{j_1, \ldots, j_r\} \subset \{1, \ldots, J_0\}$, $\nu_u \in \mathbb{N}$, $1 \leq u \leq r$. We then define the <u>box</u>

$$\mathcal{B}(m_0, j_1, \ldots, j_r, \nu_1, \ldots, \nu_r) = \{n \colon m_0(n) = m_0,\ \nu(n, I_{j_u}) = \nu_u \text{ for } 1 \leq u \leq r,\ \nu(n, I_j) = 0 \text{ for } j \notin \{j_1, \ldots, j_r\}\}.$$

We also use the vectors notations

$$\vec{j} = (j_1, \ldots, j_r) \quad \text{and} \quad \vec{\nu} = (\nu_1, \ldots, \nu_r)$$

and write $\mathcal{B}(m_0, \vec{j}, \vec{\nu})$.
Obviously each n belongs to at most one box, each $n \leq x$ to exactly one box which we denote by $\mathcal{B}(n)$.

<u>Definition 4.2</u> (The mollifier):
We set

$$\tilde{\mu}(m) = \begin{cases} \mu(m), & \text{if,} \quad m \in \mathcal{X} \text{ or } m = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathcal{X}$ is the union of all boxes $\mathcal{B}(m_0, y_{j_1}, \ldots, y_{j_r}, \nu_1, \ldots, \nu_r)$ with

$$m_0 y_{j_1}^{\nu_1} \cdots y_{j_r}^{\nu_r} \leq x. \tag{3}$$

We define

$$M(s, \chi) = \sum_m \tilde{\mu}(m) \chi(m) m^{-s}.$$

<u>Lemma 4.1</u>:
For $1 \leq x$ we have

$$\sum_{m|l} \tilde{\mu}(m) = 0.$$

Proof:
From $1 \leq l \leq x$ and $m|l$ it follows that $m = 1$ or that the box $\mathcal{B}(m)$ satifies (3) and thus $\tilde{\mu}(m) = \mu(m)$ for all $m|l$. Thus

$$\sum_{m|l} \tilde{\mu}(m) = \sum_{m|l} \mu(m) = 0.$$

<u>Definition 4.3:</u>
For a box $B(m_0, \vec{j}, \vec{\nu})$, $q \in \mathbb{N}$ and $s \in \mathbb{C}$ we set

$$\begin{aligned} \sum\nolimits_1 (\mathcal{B}, q, s) &= \sum_{n \colon \mathcal{B}(n) = \mathcal{B}} \chi(nq^2)(nq^2)^{-s} \\ \sum\nolimits_{1,b} (\mathcal{B}, q, s) &= \sum_{\substack{n \colon \mathcal{B}(n) = \mathcal{B} \\ q^2 n \leq x}} \chi(nq^2)(nq^2)^{-s} \\ \sum\nolimits_{(\mu)} (\mathcal{B}, s) &= \sum_{m \colon \mathcal{B}(m) = \mathcal{B}} \tilde{\mu}(m) \chi(m) m^{-s}. \end{aligned}$$

In the sequal we want to prove that $L(\sigma_3 + it_3, \chi) \neq 0$ for all $\sigma_3 = \sigma_0 + 10\epsilon$ for sufficiently large B, where $\epsilon = \epsilon(B)$ is any function with $\lim_{B \to \infty} \epsilon(B) = 0$.

Definition 4.4:
We set $\sigma_1 = \sigma_0 + \epsilon$ and $\sigma_2 = \sigma_0 + 2\epsilon$.

5 Cutoff by complex integration

$\sum_{1,b}$ is obtained from $\sum_1$ by adding the condition $q^2 n \leq x$. This cutoff may be accomplished by complex integration.

Lemma 5.1 (Perron's formula):
Let $c > 0$, $T > 0$ and $q > 0$. Then we have for $T \to \infty$

$$\int_{c-iT}^{c+iT} \frac{y^s}{s}\, ds = \begin{cases} 1 + O\left(\frac{y^c}{T|\log y|}\right), & \text{if } y > 1 \\ O\left(\frac{y^c}{T|\log y|}\right), & \text{if } 0 < y < 1. \end{cases}$$

Proof: see [].

Lemma 5.2:
Let $\mathcal{B} = \mathcal{B}(n_0, \vec{j}, \vec{\nu})$, $s_1 = \sigma_1 + it_1$. For $T \geq 1$ we have

$$\sum\nolimits_{1,b}(\mathcal{B}, q, s) = \frac{1}{2\pi i} \int_{\epsilon-iT}^{\epsilon+iT} \sum\nolimits_1 (\mathcal{B}, q, s_1 + s) \frac{x^s}{s}\, ds + O\left(\left(\frac{x}{q^4 n_0} \cdot 2^{\nu_1 + \ldots + \nu_r}\right)^{1-\sigma_0} \frac{n_0^{-2\sigma_0} q^{-4\sigma_0}}{T}\right).$$

Proof: We apply Lemma 5.1 with $c = \epsilon$ and obtain

$$\sum\nolimits_{1,b}(\mathcal{B}, q, s) = \frac{1}{2\pi i} \int_{\epsilon-iT}^{\epsilon+iT} \sum\nolimits_1 (\mathcal{B}, q, s_1 + s) \frac{x^s}{s}\, ds + O\left(\frac{1}{T}\left(\left|\sum\nolimits^{(1)}\right| + \left|\sum\nolimits^{(2)}\right| + \left|\sum\nolimits^{(3)}\right|\right)\right)$$

with

$$\begin{aligned} \left|\sum\nolimits^{(1)}\right| &= \sum_{n < \frac{1}{2}\frac{x}{q^2 n_0}} \frac{n^{-\sigma_2}}{\left|\log \frac{x}{q^2 n_0 n}\right|} \\ \left|\sum\nolimits^{(2)}\right| &= \sum_{\frac{1}{2}\frac{x}{q^2 n_0} \leq n < \frac{2x}{q^2 n_0}} \frac{n^{-\sigma_2}}{\left|\log \frac{x}{q^2 n_0 n}\right|} \\ \left|\sum\nolimits^{(3)}\right| &= \sum_{\frac{2x}{q^2 n_0} < n \leq \frac{x}{q^2 n_0} \cdot 2^{\nu_1 + \ldots + \nu_r}} \frac{n^{-\sigma_2}}{\left|\log \frac{x}{q^2 n_0 n}\right|}. \end{aligned}$$

In $\sum^{(1)}$ and $\sum^{(3)}$ we have $\left|\log \frac{x}{q^2 n_0 n}\right|^{-1} = O(1)$ and thus

$$\begin{aligned} \left|\sum\nolimits^{(1)}\right| &\ll \sum_{n < \frac{1}{2}\frac{x}{q^2 n_0}} n^{-\sigma_2} \ll \int_1^{\frac{x}{2n_0}} u^{-\sigma_2}\, du \ll_{\sigma_0} \left(\frac{x}{n_0}\right)^{1-\sigma_2} \\ \left|\sum\nolimits^{(3)}\right| &\ll \left(\frac{x}{q^2 n_0} \cdot 2^{\nu_1 + \ldots + \nu_r}\right)^{1-\sigma_2}. \end{aligned}$$

Estimate of $\sum^{(2)}$:
Let L be an integer clostest to $\frac{x}{q^2 n_0}$. For $L < n \leq 2x$ let $r = n - L$. Then, since $\frac{x}{q^2 n_0} \leq L + \frac{1}{2}$ we have the estimate

$$\log\left(\frac{n_0 n q^2}{x}\right) \geq \log\left(\frac{L + r}{L + \frac{1}{2}}\right) = \log\left(1 + \frac{r - \frac{1}{2}}{L + \frac{1}{2}}\right).$$

In the sequal let $c_0, c_1 > 0$ be fixed constants. From the mean- value Theorem we have that $\log(1+u) \geq c_0 u$ for $0 \leq u \leq 1$ and obtain

$$\log\left(1 + \frac{r-\frac{1}{2}}{L+\frac{1}{2}}\right) \geq c_0 \frac{r-\frac{1}{2}}{L+\frac{1}{2}} \geq c_1 \frac{rq^2 n_0}{x}.$$

Thus

$$\sum{}^{(2)} = \sum_{\frac{1}{2}\frac{x}{q^2 n_0} \leq n < \frac{2x}{q^2 n_0}} \frac{1}{q^2 n^{\sigma_2} \left|\log \frac{x}{n_0 n}\right|} = O\left(\left(\frac{x}{q^2 n_0}\right)^{1-\sigma_1} \sum_{1 \leq r \leq \frac{2x}{q^2 n_0}} \frac{1}{r}\right) \ll \left(\frac{x}{q^2 n_0}\right)^{1-\sigma_1} \log\left(\frac{2x}{q^2 n_0}\right).$$

This concludes the proof of lemma 5.2.

We expect the mollifier $M(s,\chi)$ to be an approximation to the reciprocal of $L(s,\chi)$. An evaluation of $L_x(s,\chi)M(s,\chi)$ by definitions 4.1 and 4.2 gives

Lemma 5.3:

$$\begin{aligned} L_x(s,\chi)M(s,\chi) &= \sum{}_{1,b}(\mathcal{B}_1, q, s) \sum{}_{(\mu)}(\mathcal{B}_2, s) = \sum_{\mathcal{B}_1,\mathcal{B}_2} \sum_{\substack{n\in\mathcal{B}_1 \\ n\leq x}} \sum_{m\in\mathcal{B}_2} \tilde{\mu}(m)\chi(mn)(mn)^{-s} \\ &= \sum_{l} \left(\sum_{\substack{m|l,\ l/m\leq x \\ m\in\mathcal{X}}} \tilde{\mu}(m) \right) l^{-s}. \end{aligned}$$

By Lemma 4.1 the inner sum is 0 for $l \leq x$. Thus by a second cutoff we may remove from each pair $(\mathcal{B}_1, \mathcal{B}_2)$ of boxes alle the pairs (m,n) for which $m \cdot n \leq x$.

Lemma 5.4:
Let $s_2 = \sigma_2 + it_2$. Let $\mathcal{B}_1 = \mathcal{B}(m_0, q, \vec{\nu}, \vec{j})$ and $\mathcal{B}_2 = \mathcal{B}(m_0, 1, \vec{\lambda}, \vec{k})$. Then we have

$$\begin{aligned} \sum_{\substack{(m,n):\ m\in\mathcal{B}_1, n\in\mathcal{B}_2 \\ m\cdot n\leq x, n\leq x}} \tilde{\mu}(m)(mn)^{-s_2} &= \frac{1}{2\pi i}\int_{\epsilon-iT}^{\epsilon+iT} \sum{}_{1,b}(\mathcal{B}_1, s_2+s) \sum{}_{(\mu)}(\mathcal{B}_2, s_2+s) \frac{x^s}{s}\, ds \\ &\quad + O\left(x^{\epsilon} \left(\frac{x \cdot 2^{\nu_1+\ldots+\nu_{r_1}+\lambda_1+\ldots+\lambda_{r_2}}}{q^4 m_0 n_0} \right)^{1-\sigma_0} \frac{(m_0 n_0)^{-2\sigma_0} q^{-4\sigma_0}}{T} \right). \end{aligned}$$

Proof: We apply Lemma 5.1 with $c = \epsilon$ and obtain

$$\begin{aligned} \sum_{\substack{(m,n):\ m\in\mathcal{B}_1, n\in\mathcal{B}_2 \\ m\cdot n\leq x, n\leq x}} \tilde{\mu}(m) &= \frac{1}{2\pi i}\int_{\epsilon-iT}^{\epsilon+iT} \sum{}_{1,b}(\mathcal{B}_1, s_2+s) \sum{}_{(\mu)}(\mathcal{B}_2, s_2+s) \frac{x^s}{s}\, ds \\ &\quad + O\left(\frac{1}{T} \left(|\sum{}^{(4)}| + |\sum{}^{(5)}| + |\sum{}^{(6)}| \right) \right) \end{aligned}$$

with

$$\sum\nolimits^{(4)} = \sum_{\substack{(m,n):\ m\cdot n\le \frac{1}{2x} \\ n<\frac{1}{2}\frac{x}{q^2}}} \frac{|mn|^{-\sigma_2}}{\left|\log \frac{x}{q^2 m_0 n_0 mn}\right|}$$

$$\sum\nolimits^{(5)} = \sum_{(m,n):\ \frac{1}{2}\frac{x}{q^2 m_0 n_0}\le m\cdot n\le \frac{2xmn}{q^2 m_0 n_0}} \frac{|mn|^{-\sigma_2}}{\left|\log \frac{x}{q^2 m_0 n_0 mn}\right|}$$

$$\sum\nolimits^{(6)} = \sum_{(m,n):\ \frac{2x}{q^2 m_0 n_0}<m\cdot n\le \frac{x}{q^2 m_0 n_0}\cdot 2^{\nu_1+\ldots\nu_{r_1}+\lambda_1+\ldots+\lambda_{r_2}}} \frac{|mn|^{-\sigma_2}}{\left|\log \frac{x}{q^2 m_0 n_0 mn}\right|}.$$

By using the wellknown upper bound for the divisor function $d(n) \ll n^\epsilon$ we obtain the claim of Lemma 5.4 in an anlogoues manner to the proof of Lemma 5.2.

Lemma 5.5:
We have

$$\begin{aligned} L_x(s,\chi)M(s,\chi) &= 1 + \sum_{(\mathcal{B}_1,\mathcal{B}_2)}{}' \Bigg(\sum_{\substack{n\in\mathcal{B}_1 \\ n\le x}} \sum_{m\in\mathcal{B}_2} \tilde{\mu}(m)\chi(mn)(mn)^{-s_2} \\ &\quad - \frac{1}{2\pi i}\int_{\epsilon-iT}^{\epsilon+iT} \sum\nolimits_{1,b}(\mathcal{B}_1, s_2+s) \sum\nolimits_{(\mu)}(\mathcal{B}_2, s_2+s)\frac{x^s}{s}\,ds \Bigg) \\ &\quad + O\left(x^\epsilon \left(\frac{x}{q^4 m_0 n_0}\cdot 2^{\nu_1+\ldots\nu_{r_1}+\lambda_1+\ldots+\lambda_{r_2}} \right)^{1-\sigma_0} \frac{(m_0 n_0)^{-2\sigma_0} q^{-4\sigma_0}}{T} \right), \end{aligned}$$

where the sum $\sum'$ is extended over all pairs $(\mathcal{B}_1,\mathcal{B}_2)$ of boxes $\mathcal{B}_1 = \mathcal{B}(n_0, q, \nu_1, \ldots, \nu_{r_1}, j_1, \ldots, j_{r_1})$, $\mathcal{B}_2 = \mathcal{B}(m_0, 1, \lambda_1, \ldots, \lambda_{r_2}, k_1, \ldots, j_{k_2})$ with

$$m_0 n_0 q^2 y_{j_1}^{\nu_1} \cdots y_{j_r}^{\nu_{r_1}} \cdot y_{k_1}^{\lambda_1} \cdots y_{k_{r_2}}^{\lambda_{r_2}} \ge x^{\frac{9}{10}}. \tag{4}$$

Proof: We have by Lemma 5.3

$$L_x(s_2,\chi)M(s_2,\chi) = 1 + \sum_{(\mathcal{B}_1,\mathcal{B}_2)} \sum_{\substack{n\in\mathcal{B}_1 \\ n\le x}} \sum_{\substack{m\in\mathcal{B}_2 \\ mn>x}} \tilde{\mu}(m)\chi(mn)(mn)^{-s_2}.$$

The inner double sum is empty, if $(\mathcal{B}_1,\mathcal{B}_2)$ does not satisfy (4). The claim of Lemma 5.5 now follows from Lemma 5.4.

Definition 5.1:
We define recursively $\log_k x$ by $\log_1 x = \log x$ and $\log_k x = \log(\log_{k-1} x)$.

Lemma 5.6:
With fixed constants $c_1, c_2 > 0$ we have:
The number of tuplets $\vec{\nu}$ is $\ll \exp\left(c_1 \frac{\log x}{\log_2 x}\log_3 x\right)$. For fixed m_0, q we have for the number of boxes

$$|\{\mathcal{B}(m_0, q, \vec{j}, \vec{\nu})\}| \ll \exp\left(c_2 \frac{\log x}{\log_2 x}\log_3 x\right).$$

Proof: We have $\log y_{j_1} + \ldots + \log y_{j_{r_1}} = l + \frac{\log 2}{j_1+\ldots+j_{r_1}}$. Thus the sum $j_1 + \ldots + j_{r_1} = J$ may assume at most $O(\log^2 x)$ values $J \in \mathbb{N}$. For fixed r_1 and $J \in \mathbb{N}$ the number of possibilities to choose the j_1 is $\binom{J+r_1-1}{r_1}$. Because of $J \ll \log x$ and $r_1 \ll \frac{\log x}{\log_2 x}$ by Stirling's formula the number of tuplets $\vec{j}$ is $\ll \exp\left(c_1 \frac{\log x}{\log_2 x} \log_3 x\right)$. Since $\sum_{r=1}^{r_1} \nu_r \leq \nu(n)$, $\sum_{r=1}^{r_1} \nu_r \ll \frac{\log x}{\log_2 x}$ the bound for the tuplets $\vec{\nu}$ follows in the same manner.

Definition 5.2:
Let $g(s_2)$ be the vertical line

$$g(s_2) := \{s_2 + it\colon t \in [-x^{1-\sigma_2+\epsilon}, x^{1-\sigma_2+\epsilon}].$$

Lemma 5.7:
We have

$$|L_x(s_3,\chi)M(s_3,\chi)-1| \ll \exp\left(c_1 \frac{\log x}{\log_2 x} \log_3 x\right) \cdot \sum_{(\mathcal{B}_1,\mathcal{B}_2)}{}' \max_{s\in g(s_2)} \left|\sum\nolimits_1 (\mathcal{B}_2, q, s^{(2)}+s)\right| \cdot \max_{s\in g(s^{1)})} \left|\sum\nolimits_{(\mu)} (\mathcal{B}_1, s^{(2)}+s)\right|$$

Proof: This follows from Lemmas 5.1 to 5.5.

6 Relation to exponential sums over primes

We now discuss the relation of the sums $\sum_1(\mathcal{B},q,s)$ and $\sum_{(\mu)}(\mathcal{B},s)$ to the sums $\sum_{M<p\leq 2M} \chi(p)p^{-it}$. We have for $\mathcal{B} = \mathcal{B}(n_0, j_1, \ldots, j_r, \nu_1, \ldots, \nu_r)$

$$\sum\nolimits_1(\mathcal{B},q,s) = \chi(n_0)n_0^{-s}\chi(q)^2 q^{-2s} \prod_{u=1}^{r} \sum{}^{(\nu_u, j_u)}$$

with

$$\sum{}^{(\nu_u, j_u)} = \sum \chi(n^{(u)})(n^{(u)})^{-s},$$

where $n^{(u)}$ runs over all numbers of the form $n^{(u)} = p_{1,j_u} \cdots p_{\nu_u,j_u}$, $p_{v,j_u} \in I_{j_u}$ with $p_{v_1,j_u} \neq p_{v_2,j_u}$ for $v_1 \neq v_2$.
In the sequal we eliminate the restriction $p_{v_1,j_u} \neq p_{v_2,j_u}$ by the inclusion- exclusion- principle. For $\vec{v}_{j_u} = (v_1, v_2)$ with $v_1, v_2 \in \{1, \ldots, v_u\}$ let $f(\vec{v}_{j_u}) = f(j_u, \nu_u, \vec{v}_{j_u})$ be the set of all tuplets $\vec{p} = (p_1, \ldots, p_{\nu_u})$ with p_v prime, $p_v \in I_{j_u}$ and $p_{v_1} = p_{v_2}$. (The p_v are not orderes by size.)

Definition 6.1:
For $\vec{p}_{j_u} = (p_{1,j_u}, \ldots, p_{\nu_u,j_u})$ we set $\prod(\vec{p}_{j_u}) = p_{1,j_u} \cdots p_{\nu_u,j_u}$. We obtain

$$\sum_{\substack{\vec{p}_{j_u}\,:\, p_{v_1,j_u} \neq p_{v_2,j_u} \\ v_1 \neq v_2,\ p_{v,j_u} \in I_{j_u}}} \chi(p_{1,j_u} \cdots p_{\nu_u,j_u})(p_{1,j_u} \cdots p_{\nu_u,j_u})^{-s} = \sum_{\vec{p}_{j_u} = (p_1,\ldots,p_{\nu_u})} \chi(p_{1,j_u} \cdots p_{\nu_u,j_u})(p_{1,j_u} \cdots p_{\nu_u,j_u})^{-s}$$
$$+ \sum_{w=1}^{\binom{\nu_u}{2}} (-1)^w \sum_{\vec{v}_{1,j_u},\ldots,\vec{v}_{w,j_u}} \sum_{p_{j_u} \in f(\vec{v}_{1},j_u) \cap \ldots \cap f(\vec{v}_w, j_u)} \chi\left(\prod(\vec{p}_{j_u})\right)\left(\prod(\vec{p}_{j_u})\right)^{-s} \qquad (5)$$

The condition $p_{j_u} \in f(\vec{v}_1, j_u) \cap \ldots \cap f(\vec{v}_w, j_u)$ is equivalent to a set of conditions of the form

$$\begin{aligned}
p_{v_1^{(1)}, j_u} &= p_{v_2^{(1)}, j_u} = \ldots = p_{v_{\kappa_1}^{(1)}, j_u},\ \mathcal{N}_1 = \{v_1^{(1)}, \ldots, v_{\kappa_1}^{(1)}\} \\
p_{v_1^{(2)}, j_u} &= p_{v_2^{(2)}, j_u} = \ldots = p_{v_{\kappa_2}^{(2)}, j_u},\ \mathcal{N}_2 = \{v_1^{(2)}, \ldots, v_{\kappa_2}^{(2)}\} \\
&\vdots \\
p_{v_1^{(\omega)}, j_u} &= p_{v_2^{(\omega)}, j_u} = \ldots = p_{v_{\kappa_\omega}^{(\omega)}, j_u},\ \mathcal{N}_\omega = \{v_1^{(\omega)}, \ldots, v_{\kappa_\omega}^{(\omega)}\}. \qquad (6)
\end{aligned}$$

This leads to
Definition 6.2:
For $\nu_u \in \mathbb{N}$ and a tuplet $\vec{\mathcal{K}}_\omega = (\kappa_1, \ldots, \kappa_\omega)$ of natural numbers $\kappa_\omega \geq 2$ with $\kappa_1 + \ldots + \kappa_\omega \leq \nu_u$ let $\mathcal{S} = (\nu_u, \kappa_\omega)$ be the set of all tuplets $\vec{\mathcal{N}} = (\mathcal{N}_1, \ldots, \mathcal{N}_\omega)$ of subsets $\mathcal{N}_\varphi \subset \{1, \ldots, \nu_u\}$ with $\mathcal{N}_{\varphi_1} \cap \mathcal{N}_{\varphi_2} = \emptyset$ for $\varphi_1 \neq \varphi_2$. $|\mathcal{N}_\varphi| = \kappa_\varphi$, $1 \leq \varphi \leq \omega$. $sgn(\vec{\mathcal{N}}) \in \{-1, 1\}$ comes from the factor $(-1)^w$ in (5) and the condition $p_{j_u} \in f(\vec{v}_1, j_u) \cap \ldots \cap f(\vec{v}_w, j_u)$ with leads to (6) and thus to the definition of $\vec{\mathcal{N}}$.

We obtain
Lemma 6.1:

$$\begin{aligned}
\nu_u! \sum\nolimits^{(\nu_u, j_u)} &= \sum_{\substack{(p_{1,j_1} \ldots, p_{v_u, j_u}) \\ v_1 \neq v_2,\ p_{v,j_u} \in I_{j_u}}} \chi(p_{1,j_u} \cdots p_{v_u, j_u})(p_{1,j_u} \cdots p_{v_u, j_u})^{-s} = \left(\sum_{p \in I_{j_u}} \chi(p) p^{-s} \right)^{\nu_u} \\
&+ \sum_{\substack{\vec{\mathcal{K}}_\omega = (\kappa_1, \ldots, \kappa_\omega) \\ \kappa_1 + \ldots + \kappa_\omega \leq \nu_u \\ \kappa_\varphi \geq 2}} \sum_{\vec{\mathcal{N}} \in S(\nu_u, \kappa_\omega)} sgn(\vec{\mathcal{N}}) \prod_{\varphi=1}^{\omega} \left(\sum_{p_\varphi \in I_{j_u}} \chi(p_\varphi^{\kappa_\varphi}) p_\varphi^{-\kappa_\varphi s} \right) \cdot \left(\sum_{p \in I_{j_u}} \chi(p) p^{-s} \right)^{\nu_u - (\kappa_1 + \ldots + \kappa_\omega)}
\end{aligned}$$

The sums $\sum_{(\mu)}$ are treated in an analogous manner.
Lemma 6.2:
We have for $\mathcal{B} = \mathcal{B}(m_0, k_1, \ldots, k_r, \lambda_1, \ldots, \lambda_r)$

$$\sum\nolimits_{(\mu)} (\mathcal{B}, s) = \mu(m_0) \chi(m_0) m_0^{-s} \prod_{u=1}^{r} \sum\nolimits_{(\mu)}^{(\lambda_u, k_u)}$$

with

$$\sum\nolimits_{(\mu)}^{(\lambda_u, k_u)} = \sum \mu(m^{(u)}) \chi(m^{(u)}) (m^{(u)})^{-s},$$

where $m^{(u)}$ runs over all numbers of the form $m^{(u)} = p_{1,k_u} \cdots p_{\lambda_u, k_u}$ and $p_{v,k_u} \in I_{k_u}$ with $p_{v_1, k_u} \neq p_{v_2, k_u}$ for $v_1 \neq v_2$.

Lemma 6.3:

$$\begin{aligned}
\nu_u! \sum\nolimits^{(\nu_u, k_u)} &= (-1)^{\lambda_u} \left(\sum_{p \in I_{k_u}} \chi(p) p^{-s} \right)^{\lambda_u} \\
&+ \sum_{\substack{\vec{\Psi}_\omega = (\psi_1, \ldots, \psi_\omega) \\ \psi_1 + \ldots + \psi_\omega \leq \lambda_u \\ \psi_\varphi \geq 2}} \sum_{\vec{\mathcal{N}} \in S(\lambda_u, \psi_\omega)} sgn(\vec{\mathcal{N}}) \prod_{\varphi=1}^{\omega} \left(\sum_{p_\varphi \in I_{k_u}} \chi(p_\varphi^{\psi_\varphi}) p_\varphi^{-\psi_\varphi s} \right) \cdot \left(\sum_{p \in I_{k_u}} \chi(p) p^{-s} \right)^{\lambda_u - (\psi_1 + \ldots + \psi_\omega)}
\end{aligned}$$

Lemma 6.4:

$$\begin{aligned}|S(\nu_u,\kappa_\omega)| &\ll \exp\left(c\frac{\log x}{\log_2 x}\log_3 x\right)\\ |S(\lambda_u,\psi_\omega)| &\ll \exp\left(c\frac{\log x}{\log_2 x}\log_3 x\right)\end{aligned}$$

for fixed $c > 0$.
Proof: By applying Stirling's formula to the multinomial coefficient.

We now carry out the substitutions

$$\begin{aligned}\zeta_u = \zeta_u(\nu_u,\kappa_1,\ldots,\kappa_\omega) &= \nu_u-\kappa_1-\ldots-\kappa_\omega \quad \text{and}\\ \vartheta_u=\vartheta_u(\lambda_u,\psi_1,\ldots,\psi_\omega) &= \lambda_u-\psi_1-\ldots-\psi_\omega\end{aligned}$$

and obtain from the lemma 5.6, 6.1, 6.3 and 6.4.

Lemma 6.5:

$$\begin{aligned}|L_x(s^{(2)},\chi)M(s^{(2)},\chi)-1| \;\leq\; & \exp\left(c\frac{\log x}{\log_2 x}\log_3 x\right)\cdot\\ & \sum_{(m_0,n_0,q,\vec{\zeta},\vec{k},\vec{\vartheta},\vec{j})}'' m_0^{-\sigma_3}n_0^{-\sigma_3}q^{-2\sigma_3}\prod_{u=1}^{r_1}\frac{1}{\zeta_u!}\max_{s\in g(s^{(2)})}\left|\sum_{p\in I_{j_u}}\chi(p)p^{-s}\right|^{\zeta_u}\cdot\\ & \prod_{u=1}^{r_2}\frac{1}{\vartheta_u!}\max_{s\in g(s^{(2)})}\left|\sum_{p\in I_{k_u}}\chi(p)p^{-s}\right|^{\vartheta_u}\left|\sum_{r\ \text{squarefree}}\vartheta(r)r^{-\sigma_3}\right|+O(x^{-\epsilon}),\end{aligned}$$

where the summation in $\sum''$ is over all septuplets with $m_0n_0q^2y_{\kappa_1}^{\nu_1}\cdots y_{\kappa_{\omega_1}}^{\nu_{\omega_1}}y_{\psi_1}^{\lambda_1}\cdots y_{\psi_{\omega_2}}^{\lambda_{\omega_2}}\geq x^{9/10}$.

7 Smooth numbers, end of the proof

We now make use of the hypothesis of Theorem 2:

$$\sum_{M\leq p<M'}\chi(p)p^{it}\leq M^{\sigma_0}$$

with $M\leq M'<2M$ for $M\geq \ell^A$ and $\sigma_0=1-\frac{1}{A}$.
From Lemma 6.4 we have

$$\begin{aligned}|L_x(s_0,\chi)M(s_0,\chi)-1| \;\leq\; & \exp\left(c\frac{\log x}{\log_2 x}\log_3 x\right)\cdot\\ & \sum_{(m_0,n_0,q,\vec{\lambda},\vec{\nu}\vec{k},\vec{j})}'' m_0^{-\sigma_3}n_0^{-\sigma_3}q^{-2\sigma_3}\prod_{u=1}^{r_1}\frac{1}{\zeta_u!}y_u^{-\epsilon\zeta_u}\prod_{u=1}^{r_2}\frac{1}{\vartheta_u!}y_{k_u}^{-\epsilon\vartheta_u}+O(x^{-\epsilon}),\end{aligned} \quad (7)$$

where the summation in $\sum''$ is over all septuplets with $m_0n_0q^2y_{j_1}^{\nu_1}\cdots y_{j_{u_1}}^{\nu_{u_1}}y_{k_1}^{\lambda_1}\cdots y_{k_{u_2}}^{\lambda_{u_2}}$.
For the sake of simplicity we treat only the case $q=1$.
We make another partition of the sum

$$\sum'' = \sum_{(M,N,R,S)}\sum(M,N,R,S),$$

where

$$\sum(M,N,R,S) = \sum_{m_0,n_0,\vec{\lambda},\vec{\nu},\vec{k},\vec{j})} \sum_{2^M<m_0y<2^{M+1}} \sum_{2^N<n_0y<2^{N+1}} m_0^{-\sigma_3} n_0^{-\sigma_3}$$
$$\sum_{R<m_0 y_{j_1}^{\nu_1}\cdots y_{j_{u_1}}^{\nu_{u_1}}\le 2R} \sum_{S<n_0 y_{k_1}^{\lambda_1}\cdots y_{k_{r_2}}^{\lambda_{r_2}}\le 2S} \prod_{u=1}^{r_1} \frac{1}{\zeta_u!} y_{j_u}^{-\epsilon\zeta_u} \prod_{u=1}^{r_2} \frac{1}{\vartheta_u!} y_{k_u}^{-\epsilon\vartheta_u}.$$

We need a result on smoothe numbers:
<u>Definition 7.1</u>: For $1 \le y \le x$ let

$$\psi(x,y) = |\{n \le x \colon p|n \Rightarrow p \le y\}| \quad \text{and} \quad u = \frac{\log x}{\log y}.$$

<u>Lemma 7.1</u>: For $y > (\log x)^{1+\epsilon}$ we have

$$\psi(x,y) \le x \exp(-u \log u(1+o(1)).$$

<u>Proof</u>: Hildebrand []
<u>case A:</u> $S \ge x^{9/20}$
case distinction for (n_0, S):
<u>case 1:</u> $N \le c_1 \frac{\log S}{\log_2 S}$:
From (7) we have

$$\sum(M,N,R,S) \ll x^{-\epsilon}. \tag{8}$$

<u>case 2:</u> $N > c_1 \frac{\log S}{\log_2 S}$:
By lemma 7.1 we obtain

$$\begin{aligned}\psi(2^{N+1},\ell) &\ll 2^N \exp\left(\frac{-N\log 2}{A\log \ell}(\log N + \log_2 \ell(1+o(1))\right)\\ \sum(M,N,R,S) &\ll 2^{N(1-\sigma_3)} \exp\left(\frac{-N\log 2}{A\log \ell}(\log N + \log_2 \ell(1+o(1))\right).\end{aligned}$$

From $\sigma_0 = 1 - \frac{1}{A}$ we obtain

$$\sum(M,N,R,S) \ll \exp\left(-c\frac{\log x}{\log_2 x}\log_3 x\right). \tag{9}$$

<u>case A:</u> $R \ge x^{9/20}$
This is treated in an anlogous manner.

From (7)-(9) we now have

$$|L_x(s,\chi)\cdot M(s,\chi) - 1| \exp\left(-c\frac{\log x}{\log_2 x}\log_3 x\right).$$

From lemma 3.1 and the bound $M(s,\chi) \ll x^\epsilon$ the claim of Theorem 2 follows.