Weighted D-T moduli revisited and applied

D. Leviatan

Tel Aviv University

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Joint work with K. Kopotun and I. A. Shevchuk

Introduction

For $1 \leq p < \infty$ and $r \in \mathbf{N_0}$, denote for $r \geq 1$,

$$\mathbf{B}_p^r := \{f : f^{(r-1)} \in AC_{loc}(-1,1) \quad \text{and} \quad \|f^{(r)}\varphi^r\|_p < +\infty\},$$

where $\varphi(x) := \sqrt{1-x^2}$, and for $r = 0$, $\mathbf{B}_p^0 := L_p[-1,1]$.

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- For $f \in \mathbf{B}_p^r$ define

$$\omega_{k,r}^{\varphi}(f^{(r)},t)_p := \sup_{0<h\le t} \|\mathcal{W}_{kh}^r(\cdot)\Delta_{h\varphi(\cdot)}^k(f^{(r)},\cdot)\|_p,$$

  where

$$\mathcal{W}_\delta(x) := \begin{cases} \big((1-x-\delta\varphi(x)/2)(1+x-\delta\varphi(x)/2)\big)^{1/2}, \\ \qquad\qquad\qquad\qquad 1 \pm x - \delta\varphi(x)/2 \in [-1,1], \\ 0, \qquad\qquad\qquad \text{otherwise.} \end{cases}$$

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- Note that

$$\omega_{k,0}^{\varphi}(f,t)_p = \omega_\varphi^k(f,t)_p \,(\text{I prefer } = \omega_k^{\varphi}(f,t)_p).$$

The K-functional

It turns out that if $f \in \mathbf{B}_p^r$, then

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$$K_{k,r}^{\varphi}(f^{(r)}, t^k)_p := \inf_{g \in \mathbf{B}_p^{k+r}} (\|(f^{(r)} - g^{(r)})\varphi^r\|_p + t^k \|g^{(k+r)}\varphi^{k+r}\|_p),$$

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  the K-functional.
- We have the following equivalence.

Theorem

If $f \in \mathbf{B}_p^r$, then

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_p \sim K_{k,r}^{\varphi}(f^{(r)}, t^k)_p.$$

Polynomial approximation in L_p

Denote

$$E_n(f)_p := \inf_{p_n \in \mathbb{P}_n} \|f - p_n\|_p,$$

where $\mathbb{P}_n$ is the set of polynomials of degree $< n$, and let c denote a constant independent of f and n.

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If $f \in \mathbf{B}_p^r$, $1 \le p < \infty$, then

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Theorem

If $f \in \mathbf{B}_p^r$, $1 \le p < \infty$, *and* $P_n \in \mathbb{P}_n$ *denotes its polynomial of best approximation in* $L_p[-1, 1]$, *then for each* $k \in \mathbb{N}$,

$$\|\varphi^{r+k} P_n^{(r+k)}\|_p \le c n^k \omega_{k,r}^{\varphi}(f^{(r)}, 1/n)_p.$$

Rewriting the latter is

$$n^{-k}\|\varphi^{r+k}P_n^{(r+k)}\|_p \le c\omega_{k,r}^{\varphi}(f^{(r)},1/n)_p,$$

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$$\|\varphi^r\left(f^{(r)} - P_n^{(r)}\right)\|_p \le \int_0^{1/n} \left(\omega_{k,r}^{\varphi}(f^{(r)},\tau)_p/\tau\right) d\tau.$$

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- Not quite "realization".

Polynomial approximation in L_p – continued

The proof is based on two results. First, a theorem that illustrates the hierarchy between the moduli of smoothness.

Theorem

If $f \in \mathbf{B}_p^{r+1}$, $r \in \mathbb{N}_0$ *and* $1 \le p < \infty$, *and* $k \ge 2$, *then*

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_p \le ct\omega_{k-1,r+1}^{\varphi}(f^{(r+1)}, t)_p.$$

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$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_p \le c t \omega_{k-1,r+1}^{\varphi}(f^{(r+1)}, t)_p.$$

- Second, the following estimates by Ditzian and Totik. The Jackson-type estimates

$$E_n(f)_p \le c\omega_k^{\varphi}(f, 1/n)_p, \quad n \ge k,$$

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$$E_n(f)_p \le c\omega_k^{\varphi}(f, 1/n)_p, \quad n \ge k,$$

- and the estimates on the derivatives of the polynomial of best approximation

$$\|\varphi^{r+k} P_n^{(r+k)}\|_p \le cn^{r+k}\omega_{k+r}^{\varphi}(f, 1/n)_p.$$

Comments

It is also known that for $f \in L_p[-1,1]$, that if $f^{(r)} \in L_p[-1,1]$ for some $r \geq 1$, then

$$E_n(f)_p \leq cn^{-r}\omega_k^\varphi(f^{(r)}, 1/n)_p, \quad n \geq k + r.$$

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- Again, this follows from the hierarchy

$$\omega_k^{\varphi}(f^{(r-1)}, t)_p \leq ct\omega_{k-1}^{\varphi}(f^{(r)}, t)_p, \quad r \geq 1.$$

(Note that we have to assume that $f^{(r)} \in L_p[-1,1]$, the DT-moduli are not defined if the function is not in $L_p[-1,1]$.)

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$$\omega_k^{\varphi}(f^{(r-1)}, t)_p \leq ct\omega_{k-1}^{\varphi}(f^{(r)}, t)_p, \quad r \geq 1.$$

  (Note that we have to assume that $f^{(r)} \in L_p[-1,1]$, the DT-moduli are not defined if the function is not in $L_p[-1,1]$.)
- We have a **sharper estimate**, and for $\mathbf{B}_p^r$ – a **wider class of functions**.

An immediate consequence is,

Corollary

If $f \in \mathbf{B}_p^r$, $r \in \mathbb{N}_0$, *and if for some* $k \geq 1$, *and* $\alpha > r$, $\omega_{k,r}^{\varphi}(f^{(r)}, t)_p = O(t^{\alpha - r})$, *then*

$$E_n(f)_p \leq c n^{-\alpha}, \quad n \geq k + r.$$

Direct and inverse theorems

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If $f \in \mathbf{B}_p^r$, $r \in \mathbb{N}_0$, *and if for some* $k \geq 1$, *and* $\alpha > r$, $\omega_{k,r}^{\varphi}(f^{(r)}, t)_p = O(t^{\alpha-r})$, *then*

$$E_n(f)_p \leq cn^{-\alpha}, \quad n \geq k+r.$$

- We have the following inverse result.

Theorem

Let $r \in \mathbb{N}_0$, $k \geq 1$ *and* $\alpha > 0$, *be such that* $r < \alpha < r+k$, *and let* $f \in L_p[-1,1]$. *If*

$$E_n(f)_p \leq Mn^{-\alpha}, \quad n \geq 1,$$

then $f \in \mathbf{B}_p^r$ *and*

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_p \leq c(M, \alpha, r)t^{\alpha-r}, \quad t > 0.$$

Inverse consequences

Let $P_{k+r} \in \mathcal{P}_{k+r}$, be the best approximation to $f \in L_p[-1,1]$, and set $F := f - P_{k+r}$. Since $\omega_{k,r}^{\varphi}(p_{k+r}^{(r)}, t)_p \equiv 0$ for $p_{k+r} \in \mathcal{P}_{k+r}$, it follows that $\omega_{k,r}^{\varphi}(f^{(r)}, t)_p = \omega_{k,r}^{\varphi}(F^{(r)}, t)_p$, $t > 0$, that $E_n(F)_p = \|F\|_p = E_{k+r}(f)_p$, $n \le k+r$, and that we have $E_n(f)_p = E_n(F)_p$, for all $n \ge k+r$.

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- Therefore, an immediate consequence is,

Corollary

Let $r \in \mathbb{N}_0$, $k \ge 1$ and $\alpha > 0$, be such that $r < \alpha < r + k$, and let $f \in L_p[-1,1]$. If

$$E_n(f)_p \le n^{-\alpha}, \quad n \ge k + r,$$

then $f \in \mathbf{B}_p^r$ and

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_p \le c(\alpha, k, r) t^{\alpha - r}, \quad t > 0.$$

Inverse consequences – continued

We have the following extension.

Corollary

Let $r \in \mathbb{N}_0$, $k \geq 1$ *and* $\alpha > 0$, *be such that* $r < \alpha < r + k$, *and let* $f \in L_p[-1, 1]$. *If*

$$E_n(f)_p \leq n^{-\alpha}, \quad n \geq N,$$

for some $N \geq k + r$, *then* $f \in \mathbf{B}_p^r$ *and*

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_p \leq c(\alpha, k, r)t^{\alpha - r} + c(N, k, r)t^k E_{k+r}(f)_p, \quad t > 0.$$

Weighted DT moduli

Let $\|\cdot\|_p := \|\cdot\|_{L_p[-1,1]}$, $1 \le p < \infty$, and let w and ϕ be such $w, \phi \sim 1$ in compacta of $(-1,1)$, and $w(x) \sim (1 \mp x)^{\gamma(\pm 1)}$ and $\phi(x) \sim (1 \mp x)^{\beta(\pm 1)}$, as $x \to \pm 1$, where $\gamma(\pm 1), \beta(\pm 1) \ge 0$.
For $k \in \mathbb{N}_0$, let

$$\Delta_h^k(f,x) := \begin{cases} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x + (i - k/2)h), & \text{if } x \pm kh/2 \in [-1,1], \\ 0, & \text{otherwise}, \end{cases}$$

be the kth symmetric difference.

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be the kth symmetric difference.

- Similarly, the kth forward and backward differences, respectively, are

$$\overrightarrow{\Delta}_h^k(f,x) := \begin{cases} \sum_{i=0}^k (-1)^{i} \binom{k}{i} f(x + ih), & \text{if } \quad [x, x+kh] \subseteq [-1,1], \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\overleftarrow{\Delta}_h^k(f,x) := \begin{cases} \sum_{i=0}^k (-1)^{i} \binom{k}{i} f(x - ih), & \text{if } \quad [x-kh, x] \subseteq [-1,1], \\ 0, & \text{otherwise}, \end{cases}$$

Weighted DT moduli – continued

Ditzian and Totik have defined the weighted ϕ-moduli of smoothness, with weight w^p, of a function f, such that $wf \in L_p[-1,1]$, by

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- $$\begin{aligned} \omega_\phi^k(f,t)_{w,p} := & \sup_{0<h\le t} \|w\Delta_{h\phi}^k f\|_{L_p[-1+t^*,1-t^{**}]} \\ & + \sup_{0<h\le t^*} \|w\overrightarrow{\Delta}_{h\phi}^k f\|_{L_p[-1,-1+12t^*]} \\ & + \sup_{0<h\le t^{**}} \|w\overleftarrow{\Delta}_{h\phi}^k f\|_{L_p[1-12t^{**},1]}, \end{aligned}$$

Weighted DT moduli – continued

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- $$\omega_\phi^k(f,t)_{w,p} := \sup_{0<h\le t} \|w\Delta_{h\phi}^k f\|_{L_p[-1+t^*,1-t^{**}]} + \sup_{0<h\le t^*} \|w\overrightarrow{\Delta}_{h\phi}^k f\|_{L_p[-1,-1+12t^*]} + \sup_{0<h\le t^{**}} \|w\overleftarrow{\Delta}_{h\phi}^k f\|_{L_p[1-12t^{**},1]},$$

- where

$$t^* := \begin{cases} A(kt)^{1/(1-\beta(-1))}, & \text{if } \beta(-1)<1 \\ 0, & \text{if } \beta(-1)\ge 1, \end{cases}$$

and, analogously,

$$t^{**} := \begin{cases} A(kt)^{1/(1-\beta(1))}, & \text{if } \beta(1)<1 \\ 0, & \text{if } \beta(1)\ge 1, \end{cases}$$

Weighted DT moduli – continued

They called the first term "the main part modulus" and denoted it

$$\Omega^k_\phi(f,t)_{w,p} := \sup_{0<h\le t} \|w\Delta^k_{h\phi} f\|_{L_p[t^*,1-t^{**}]}.$$

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$$\Omega_{\phi}^{k}(f,t)_{w,p} := \sup_{0<h\le t} \|w\Delta_{h\phi}^{k} f\|_{L_p[t^*,1-t^{**}]}.$$

- They proved that the K-functional

$$K_{k,\phi}(f,t^k)_{w,p} := \inf_{g^{(k-1)}\in AC_{(loc)}} \left(\|(f-g)w\|_p + t^k\|w\phi^k g^{(k)}\|_p\right),$$

  is equivalent to $\omega_{\phi}^{k}(f,t)_{w,p}$.

Special weighted DT moduli

If we take $\phi(x) = \varphi(x) := \sqrt{1-x^2}$ and $w(x) = \varphi^r(x)$, the weighted DT moduli become,

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- By the above we conclude that

Theorem

If $f \in \mathbf{B}_p^r$, *then*

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_p \sim \omega_{\varphi}^{k}(f^{(r)}, t)_{\varphi^r,p}.$$

Comparing definitions

Recall that

$$\omega_{k,r}^{\varphi}(f^{(r)},t)_p := \sup_{0<h\le t} \|\mathcal{W}_{kh}^{r}(\cdot)\Delta_{h\varphi(\cdot)}^{k}(f^{(r)},\cdot)\|_p,$$

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- And compare with: take $t^* := 2k^2t^2$ and A is an absolute constant (for example, $A = 12$),

$$\omega^k_\varphi(f^{(r)},t)_{\varphi^r,p} = \sup_{0<h\le t} \|\varphi^r\Delta^k_{h\varphi}f^{(r)}\|_{L_p[-1+t^*,1-t^*]} + \sup_{0<h\le t^*} \|\varphi^r\overrightarrow{\Delta}^k_h f^{(r)}\|_{L_p[-1,-1+At^*_0]} + \sup_{0<h\le t^*} \|\varphi^r\overleftarrow{\Delta}^k_h f^{(r)}\|_{L_p[1-At^*,1]},$$

Weighted polynomial approximation

Ditzian and Totik were interested in the degree of weighted polynomial approximation (with weight w^p), and proved that for f such that $wf \in L_p[-1,1]$, we have

$$E_n(f)_{w,p} \le c \int_0^{1/n} (\Omega_\phi^r(f,\tau)_{w,p}/\tau)\, d\tau, \quad n \ge r.$$

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- They could not show that
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- Neither can we!

Weighted polynomial approximation – continued

But we can show,

Theorem

Let $0 < l < r$*, and assume that* $f^{(l-1)}$ *is locally absolutely continuous in* $(-1,1)$ *and* $w\phi^l f^{(l)} \in L_p[-1,1]$*. Then*

$$E_n(f)_{w,p} \le cn^{-l}\omega_\phi^{r-l}(f^{(l)},1/n)_{w\phi^l,p}, \quad n \ge r.$$

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But we can show,

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- Hence, in particular, we have

Theorem

Let $0 < l < r < \alpha$*, and assume that* $f^{(l-1)}$ *is locally absolutely continuous in* $(-1,1)$ *and* $w\phi^l f^{(l)} \in L_p[-1,1]$*, and* $\omega_\phi^{r-l}(f^{(l)}, t)_{w\phi^l,p} = O(t^{\alpha-l})$*, then*

$$E_n(f)_{w,p} \le cn^{-\alpha}, \quad n \ge r.$$

Inverse result

Conversely,

Theorem

Let $r \in \mathbb{N}_0$, $k \in \mathbb{N}$ *and* $r < \alpha < r + k$, *and let* f *be such that* $wf \in L_p[-1, 1]$. *If*

$$(*) \qquad E_n(f)_{w,p} \le M n^{-\alpha}, \quad n \ge 1,$$

then $f^{(r-1)}$ *is locally absolutely continuous in* $(-1, 1)$ *and*

$$(**) \qquad \omega_\phi^k(f^{(r)}, t)_{w\phi^r,p} \le c(M, l, \alpha) t^{\alpha - r}, \quad t > 0.$$

Remark

It follows by Ditzian and Totik that (*) implies,

$$\omega_{\phi}^{r+k}(f,t)_{w,p} \le c(M,\alpha)t^{\alpha}, \quad t>0.$$

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$$\omega_{\phi}^{r+k}(f,t)_{w,p} \le c(M,\alpha)t^{\alpha}, \quad t>0.$$

- However, we have no way to conclude (**) from this, as we have the hierarchy,

$$\omega_{\phi}^{r+k}(f,t)_{w,p} \le ct^{r}\omega_{\phi}^{k}(f^{(r)},t)_{w\phi^{r},p}, \quad t>0,$$

(in the wrong direction).

Concluding corollary

Corollary

Let $r \in \mathbb{N}_0$, $k \in \mathbb{N}$ *and* $\alpha > 0$, *be such that* $r < \alpha < r + k$. *Assume* w *is a weight as above, and let* $wf \in L_p[-1, 1]$. *If*

$$E_n(f)_{w,p} \le n^{-\alpha}, \quad n \ge N,$$

for some $N \ge k + r$, *then*

$$\omega_\phi^k(f^{(r)}, t)_{w\phi^r,p} \le c(w, \alpha, k, r)t^{\alpha-r} + c(w, N, k, r)t^k E_{k+r}(f)_{w,p}, \quad t > 0.$$

Moreover, if $N = k + r$, *then*

$$\omega_\phi^k(f^{(r)}, t)_{w\phi^r,p} \le c(w, \alpha, k, r)t^{\alpha-r}, \quad t > 0.$$

THANK YOU