The range of tree-indexed random walk

Jean-François Le Gall, Shen Lin

Institut universitaire de France et Université Paris-Sud Orsay

Erdös Centennial Conference
July 2013

1. Introduction

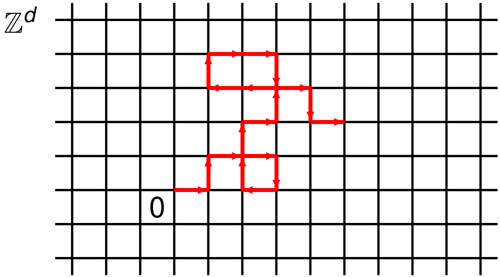
Let $(X_n)_{n\geq 0}$ be a random walk in $\mathbb{Z}^d$:

$$X_n = Y_1 + Y_2 + \cdots + Y_n$$

where $Y_1, Y_2, \ldots$ are independent and identically distributed with distribution μ.

The range $\mathbf{R}_n$ is the number of distinct sites of the lattice visited by the random walk up to time n:

$$\mathbf{R}_n := \#\{X_0, X_1, \ldots, X_n\}.$$



Here $\mathbf{R}_{19} = 17$

The Dvoretzky-Erdös asymptotics

An important result of Dvoretzky and Erdös in 1951 gives the asymptotics of R_n when $n \to \infty$, in the case of simple random walk (that is, if μ is uniform over neighbors of 0):

- if $d \geq 3$,

$$\frac{1}{n} \mathbf{R}_n \xrightarrow[n\to\infty]{\text{a.s.}} q_d > 0\,,$$

- if $d = 2$,

$$\frac{\log n}{n} \mathbf{R}_n \xrightarrow[n\to\infty]{\text{a.s.}} \pi\,,$$

- if $d = 1$,

$$n^{-1/2} \mathbf{R}_n \xrightarrow[n\to\infty]{\text{(d)}} \sup_{0\leq t\leq 1} B_t - \inf_{0\leq t\leq 1} B_t\,,$$

where q_d is the probability that X never returns to its starting point, and $(B_t)_{t\geq 0}$ is a standard linear Brownian motion.

(Dvoretzky and Erdös point out that the method extends to the case when μ is centered with finite second moments)

Applying Kingman's subadditive ergodic theorem

The method of Dvoretzky and Erdös relies on estimating the first and second moment of $\mathbf{R}_n$.

When $d \geq 3$, a quicker proof follows from Kingman's subadditive ergodic theorem. Note that, for every $m, n \geq 0$,

$$\mathbf{R}_{n+m} \leq \mathbf{R}_n + \mathbf{R}_m \circ \theta_n$$

where θ_n is the usual shift on trajectories : $X_k \circ \theta_n = X_{n+k} - X_n$

(the number of sites visited between 0 and $n+m$ is smaller than the number visited between 0 and n plus the number visited between n and $n+m$)

Kingman's theorem then gives immediately

$$\frac{1}{n}\mathbf{R}_n \xrightarrow[n\to\infty]{\text{a.s.}} q$$

where $q = \lim_{n\to\infty} \frac{1}{n} E[\mathbf{R}_n] = P(\text{no return to } 0)$

This applies to any random walk, and $q > 0$ iff X transient

Tree-indexed random walk

Question (Itai Benjamini): What is the analog of the Dvoretzky-Erdös asymptotics for a tree-indexed random walk?

Consider

- a (random) discrete rooted tree $\mathcal{T}_n$ with n vertices;
- conditionally on $\mathcal{T}_n$, a collection $(Y_e)_{e\in\mathcal{E}(\mathcal{T})}$ of independent r.v. distributed according to μ, indexed by the set of edges of $\mathcal{T}_n$.

For every vertex v of $\mathcal{T}_n$, set

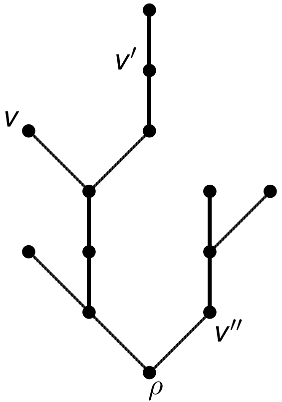
$$X_v = \sum_{e:\rho\longrightarrow v} Y_e$$

where the sum is over all edges on the path from the root ρ to v.

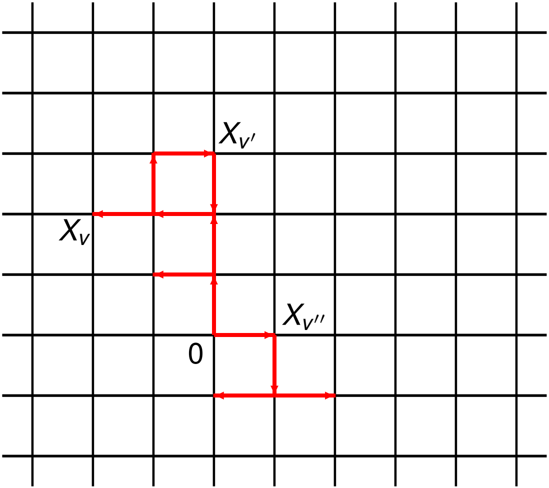
The range is $\mathcal{R}_n := \#\{X_v : v \text{ vertex of } \mathcal{T}_n\}$.

One expects $\mathcal{R}_n$ to be smaller than $\mathbf{R}_n$ (range of ordinary RW), because there are more self-intersections.

Tree-indexed random walk



Tree $\mathcal{T}_n$



Galton-Watson trees

Let θ be a probability measure on $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ such that:

- $\sum_{k=0}^{\infty} k\,\theta(k) = 1$ (criticality)
- $\sum_{k=0}^{\infty} k^2\,\theta(k) < \infty$ (finite variance)

The Galton-Watson tree $\mathcal{T}$ with offspring distribution θ describes the genealogy of a Galton-Watson branching process with offspring distribution θ:

- the process starts with 1 ancestor at generation 0;
- each individual has k children with probability $\theta(k)$.

$\longrightarrow$ can be viewed as a rooted ordered tree (put an order on the children of each individual).

θ critical $\Rightarrow \mathcal{T}$ is finite a.s. Notation: $\#\mathcal{T}$ is the number of vertices of $\mathcal{T}$

Galton-Watson trees

Let θ be a probability measure on $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ such that:

- $\displaystyle\sum_{k=0}^{\infty} k\,\theta(k) = 1$ (criticality)
- $\displaystyle\sum_{k=0}^{\infty} k^2\,\theta(k) < \infty$ (finite variance)

The Galton-Watson tree $\mathcal{T}$ with offspring distribution θ describes the genealogy of a Galton-Watson branching process with offspring distribution θ:

- the process starts with 1 ancestor at generation 0;
- each individual has k children with probability $\theta(k)$.

$\longrightarrow$ can be viewed as a rooted ordered tree (put an order on the children of each individual).

θ critical $\Rightarrow \mathcal{T}$ is finite a.s. Notation: $\#\mathcal{T}$ is the number of vertices of $\mathcal{T}$

Galton-Watson trees with a fixed progeny

Let $\mathcal{T}$ be a θ-Galton-Watson tree,
For every $n \geq 1$ such that $P(\#\mathcal{T} = n) > 0$, let

$$\mathcal{T}_n \stackrel{\text{(d)}}{=} \mathcal{T} \text{ conditioned on } \#\mathcal{T} = n$$

Then $\mathcal{T}_n$ is a random tree with n vertices.

This setting includes many "combinatorial trees" (meaning that $\mathcal{T}_n$ is then uniformly distributed on a certain class of discrete trees):

- $\theta(k) = 2^{-k-1}$: $\mathcal{T}_n$ is uniform in the class of rooted ordered trees with n vertices;
- $\theta(0) = \theta(2) = \frac{1}{2}$: $\mathcal{T}_n$ is uniform in the class of binary trees with n vertices;
- θ Poisson : $\mathcal{T}_n$ is uniform in the class of Cayley trees with n vertices.

Galton-Watson trees with a fixed progeny

Let $\mathcal{T}$ be a θ-Galton-Watson tree,
For every $n \geq 1$ such that $P(\#\mathcal{T} = n) > 0$, let

$$\mathcal{T}_n \overset{\text{(d)}}{=} \mathcal{T} \text{ conditioned on } \#\mathcal{T} = n$$

Then $\mathcal{T}_n$ is a random tree with n vertices.

This setting includes many "combinatorial trees" (meaning that $\mathcal{T}_n$ is then uniformly distributed on a certain class of discrete trees):

- $\theta(k) = 2^{-k-1}$: $\mathcal{T}_n$ is uniform in the class of rooted ordered trees with n vertices;
- $\theta(0) = \theta(2) = \frac{1}{2}$: $\mathcal{T}_n$ is uniform in the class of binary trees with n vertices;
- θ Poisson : $\mathcal{T}_n$ is uniform in the class of Cayley trees with n vertices.

The range of tree-indexed random walk

- $\mathcal{T}_n$ is a θ-Galton-Watson tree conditioned to have n vertices;
- conditionally on $\mathcal{T}_n$, $(X_v)_{v\in V(\mathcal{T}_n)}$ is random walk with jump distribution μ indexed by $\mathcal{T}_n$, and $\mathcal{R}_n = \#\{X_v : v \in V(\mathcal{T}_n)\}$.

Theorem

Suppose μ is centered with sufficiently high moments:

1. *if $d \geq 5$,* $$\frac{1}{n}\mathcal{R}_n \xrightarrow[n\to\infty]{\text{(P)}} c_{\mu,\theta} > 0$$
2. *if $d = 4$, and $\theta(k) = 2^{-k-1}$,* $$\frac{\log n}{n}\mathcal{R}_n \xrightarrow[n\to\infty]{\text{(P)}} 8\,\pi^2\,\sigma^4\,,$$ *where $\sigma^2 = (\det(\mathrm{cov}(\mu)))^{1/4}$*
3. *if $d \leq 3$,* $$n^{-d/4}\,\mathcal{R}_n \xrightarrow[n\to\infty]{\text{(d)}} c_{\mu,\theta}\,\mathrm{Leb}(\mathrm{supp}(\mathcal{I}))$$ *where $\mathcal{I}$ is ISE (Integrated Super-Brownian Excursion).*

The range of tree-indexed random walk

- $\mathcal{T}_n$ is a θ-Galton-Watson tree conditioned to have n vertices;
- conditionally on $\mathcal{T}_n$, $(X_v)_{v\in V(\mathcal{T}_n)}$ is random walk with jump distribution μ indexed by $\mathcal{T}_n$, and $\mathcal{R}_n = \#\{X_v : v \in V(\mathcal{T}_n)\}$.

Theorem

Suppose μ is centered with sufficiently high moments:

1. *if $d \geq 5$,* $$\frac{1}{n}\mathcal{R}_n \xrightarrow[n\to\infty]{(\mathrm{P})} c_{\mu,\theta} > 0$$
2. *if $d = 4$, and $\theta(k) = 2^{-k-1}$,* $$\frac{\log n}{n}\mathcal{R}_n \xrightarrow[n\to\infty]{(\mathrm{P})} 8\,\pi^2\,\sigma^4\,,$$ *where $\sigma^2 = (\det(\mathrm{cov}(\mu)))^{1/4}$*
3. *if $d \leq 3$,* $$n^{-d/4}\mathcal{R}_n \xrightarrow[n\to\infty]{(\mathrm{d})} c_{\mu,\theta}\,\mathrm{Leb}(\mathrm{supp}(\mathcal{I}))$$ *where $\mathcal{I}$ is ISE (Integrated Super-Brownian Excursion).*

Remarks on the main theorem

- Results are similar to those for ordinary random walk, BUT

  the critical dimension is now $d = 4$

  Note that $\max\{|X_v| : v \text{ vertex of } \mathcal{T}_n\} \sim n^{1/4}$ (Janson-Marckert)
- Above the critical dimension, the range grows linearly (can again be viewed as a consequence of Kingman's theorem, but this is less immediate!)
- At the critical dimension $d = 4$, the proof is more involved (our method only works for θ geometric)
- Below the critical dimension, the result is related to the "invariance principles" connecting branching random walk with super-Brownian motion
- The convergence of $\frac{1}{n}\mathcal{R}_n$ to a constant $c_{\mu,\theta} \geq 0$ extends to much more general θ and μ.

Remarks on the main theorem

- Results are similar to those for ordinary random walk, BUT

  the critical dimension is now $d = 4$

  Note that $\max\{|X_v| : v \text{ vertex of } \mathcal{T}_n\} \sim n^{1/4}$ (Janson-Marckert)
- Above the critical dimension, the range grows linearly (can again be viewed as a consequence of Kingman's theorem, but this is less immediate!)
- At the critical dimension $d = 4$, the proof is more involved (our method only works for θ geometric)
- Below the critical dimension, the result is related to the "invariance principles" connecting branching random walk with super-Brownian motion
- The convergence of $\frac{1}{n}\mathcal{R}_n$ to a constant $c_{\mu,\theta} \geq 0$ extends to much more general θ and μ.

2. The linear growth of $\mathcal{R}_n$

KEY IDEA: Apply Kingman's ergodic theorem

BUT: needs to find a suitable shift transformation on an appropriate space of trees, and a corresponding invariant probability measure.

The space of trees $\mathbb{T}$: consists of infinite trees $\mathcal{T}$ having

- a "spine" with infinitely many vertices labeled $0, -1, -2, \ldots$
- attached to each vertex $-k$ of the spine, a finite rooted ordered tree $\mathcal{T}_k$

We assume that there are infinitely many vertices not on the spine.

2. The linear growth of $\mathcal{R}_n$

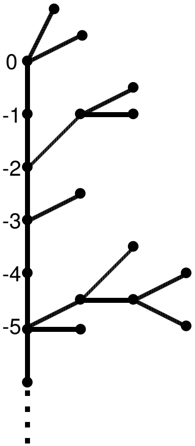
KEY IDEA: Apply Kingman's ergodic theorem

BUT: needs to find a suitable shift transformation on an appropriate space of trees, and a corresponding invariant probability measure.

The space of trees $\mathbb{T}$: consists of infinite trees $\mathcal{T}$ having

- a "spine" with infinitely many vertices labeled $0, -1, -2, \ldots$
- attached to each vertex $-k$ of the spine, a finite rooted ordered tree $\mathcal{T}_k$

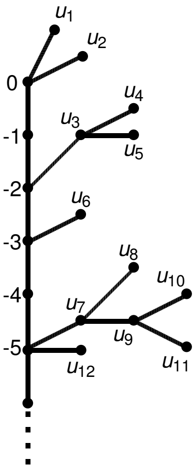
We assume that there are infinitely many vertices not on the spine.



The shift on infinite trees

Let $\mathcal{T} \in \mathbb{T}$ be an infinite tree, let $u_1, u_2, \ldots$ be the vertices of $\mathcal{T}$ not belonging to the spine enumerated in lexicographical order (considering successively the subtrees $\mathcal{T}_0, \mathcal{T}_1, \ldots$ in this order)

Define the shift $\tau(\mathcal{T})$ by declaring that the top of the spine of $\tau(\mathcal{T})$ is u_1 and removing the vertices of the spine of $\mathcal{T}$ that are not ancestors of u_1 (i.e. the vertices $0, -1, \ldots, -k+1$, if u_1 lies in $\mathcal{T}_k$)



The shift on infinite trees

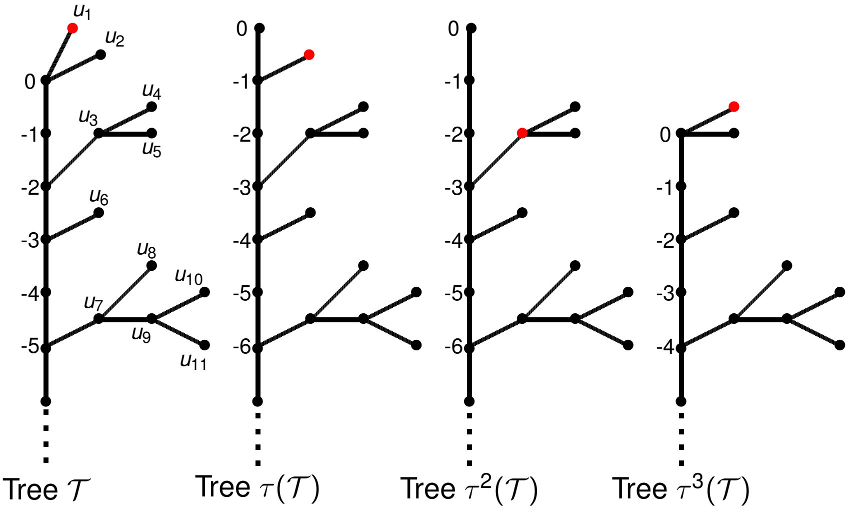


Illustration of the shift: The red vertex (the first one not on the spine) becomes the top of the spine when applying the shift.

The invariant measure

Define a probability measure $\mathbf{P}_\theta$ on $\mathbb{T}$: Under $\mathbf{P}_\theta$,

- the subtree attached at the top of the spine is a θ-Galton-Watson tree;
- each vertex of the spine other than 0 has k children (not on the spine) with probability

$$\theta([k+1,\infty))$$

- these children, and their descendants, then reproduce according to the offspring distribution θ.

Proposition

The probability measure $\mathbf{P}_\theta$ is invariant (and ergodic) under the shift τ

The proof is easy by a direct verification.

Remark. This result only requires the fact that θ is critical.

The invariant measure

Define a probability measure $\mathbf{P}_\theta$ on $\mathbb{T}$: Under $\mathbf{P}_\theta$,

- the subtree attached at the top of the spine is a θ-Galton-Watson tree;
- each vertex of the spine other than 0 has k children (not on the spine) with probability
$$\theta([k+1,\infty))$$
- these children, and their descendants, then reproduce according to the offspring distribution θ.

Proposition

The probability measure $\mathbf{P}_\theta$ is invariant (and ergodic) under the shift τ

The proof is easy by a direct verification.

Remark. This result only requires the fact that θ is critical.

Trees with spatial positions

If $\mathcal{T} \in \mathbb{T}$ is an infinite tree, define the μ-random walk $(X_v)_{v \in V(\mathcal{T})}$ indexed by $\mathcal{T}$ by imposing that:

- the increments along edges (oriented from the "bottom" of the spine) are independent and distributed according to μ
- the spatial position X_0 at the top of the spine is 0

Fact

The law of the pair consisting of a tree distributed according to $\mathbf{P}_\theta$ and the associated μ-random walk is again invariant under the shift τ.

(To define spatial positions of the shifted tree $\tau(\mathcal{T})$ one translates the spatial positions of the corresponding vertices of $\mathcal{T}$ so that the position of the top of the spine is again 0 in $\tau(\mathcal{T})$)

Linear growth of the range

Assumptions:

- $\mathcal{T}$ is distributed according to $\mathbf{P}_\theta$;
- conditionally on $\mathcal{T}$, $(X_v)_{v\in\mathcal{T}}$ is the μ-random walk indexed by $\mathcal{T}$.

Let $u_1, u_2, \ldots$ be the vertices of $\mathcal{T}$ not on the spine, enumerated in lexicographical order, and

$$R_n = \#\{X_{u_1}, X_{u_2}, \ldots, X_{u_n}\}$$

Theorem

There exists a constant $c_{\mu,\theta} \in [0,1]$ such that

$$\frac{1}{n} R_n \xrightarrow[n\to\infty]{\text{a.s.}} c_{\mu,\theta}$$

Proof. Just apply Kingman's subadditive ergodic theorem!

Remark. The theorem requires no assumption on μ or θ, except the fact that the offspring distribution θ is critical.

Positivity of the limiting constant

Assume that the random walk $(S_j)_{j\geq 0}$ with jump distribution μ is transient (otherwise $c_{\mu,\theta} = 0$). Let :

- G_μ Green function of S
- g_θ generating function of θ

Proposition

Assume that

$$\prod_{j=1}^{\infty} \Big(\frac{1 - g_\theta((1 - G_\mu(S_j))_+)}{G_\mu(S_j)} \Big) > 0 \quad a.s.$$

Then $c_{\mu,\theta} > 0$.

Corollary

Suppose that μ is centered and has finite moments of order $d - 1$.

- *If θ has finite variance, $c_{\mu,\theta} > 0$ if $d \geq 5$,*
- *If θ is in the domain of attraction of a stable distribution with index $\alpha \in (1, 2)$, then $c_{\mu,\theta} > 0$ if $d > \frac{2\alpha}{\alpha-1}$.*

From the infinite tree to trees with a given size

Need an argument to derive the result for Galton-Watson trees with a fixed progeny from the case of the infinite tree.

From an infinite tree distributed according to $\mathbf{P}_\theta$ can obtain a sequence $\mathcal{T}_{(1)}, \mathcal{T}_{(2)}, \ldots$ of independent θ-Galton-Watson trees:
$\longrightarrow$ consider the subtrees branching off the children of the vertices of the spine.

If $k_n = \min\{j : \#(\mathcal{T}_{(j)}) \geq n\}$, then $\mathcal{T}_{(k_n)}$ is a θ-Galton-Watson tree conditioned to have at least n vertices.

- Can derive from the theorem an analogous result for a θ-Galton-Watson tree conditioned to have at least n vertices. (this requires limit theorems for the contour of a sequence of independent GW trees, cf Duquesne-LG)
- An absolute continuity argument allows one to deal with a θ-Galton-Watson tree conditioned to have exactly n vertices

3. The critical dimension $d = 4$

Assumptions.

- $\theta(k) = 2^{-k-1}$ (geometric).
- μ is symmetric and has small exponential moments.

Set $\sigma^2 = (\det(\mathrm{cov}(\mu)))^{1/4}$.

IDEA: Use the path-valued Markov chain called the discrete snake to generate the spatial positions of an infinite tree distributed according to $\mathbf{P}_\theta$.

$\longrightarrow$ Then exploit the Markovian properties of the discrete snake to derive the needed estimates.

(If θ is not geometric, the discrete snake approach does not work and things become more complicated!)

The discrete snake

The discrete snake is a Markov chain with values in the set $\mathcal{W}$ of all semi-infinite discrete paths

$$w : (-\infty, \zeta] \cap \mathbb{Z} \longrightarrow \mathbb{Z}^4$$

where $\zeta = \zeta_{(w)} \in \mathbb{Z}$ is called the lifetime of w.

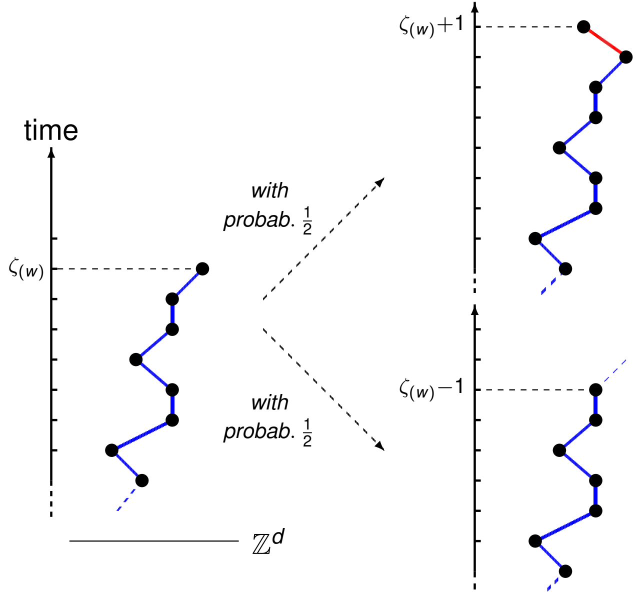
Transition kernel. Suppose that $W_0 = w$:

- With probability $\frac{1}{2}$,
  - $\zeta_{(W_1)} = \zeta_{(w)} - 1$,
  - $W_1(k) = w(k)$ for all $k \leq \zeta_{(w)} - 1$.

  (the last step of w is removed)
- With probability $\frac{1}{2}$,
  - $\zeta_{(W_1)} = \zeta_{(w)} + 1$
  - $W_1(k) = w(k)$ for all $k \leq \zeta_{(w)}$
  - $W_1(\zeta_{(w)} + 1) - W_1(\zeta_{(w)})$ has law μ

  (one step is added to w using the jump distribution μ)

Transition kernel of the discrete snake



A key estimate

Suppose that $\zeta_{(W_0)} = 0$ and $(W_0(-k))_{k\geq 0}$ is distributed as a random walk with jump distribution μ started from 0.

Write $\widehat{W}_k := W_k(\zeta_{(W_k)})$ for the terminal point of W_k (the "head of the discrete snake")

Lemma

We have

$$\lim_{n\to\infty} (\log n)\, P(\widehat{W}_k \neq \widehat{W}_0 \text{ for all } k = 1, 2, \ldots, n) = 4\pi^2\sigma^4.$$

Remark. Analogous result for a (centered, finite variance) random walk S on $\mathbb{Z}^2$ started from 0,

$$\lim_{n\to\infty} (\log n)\, P(S_k \neq 0 \text{ for all } k = 1, 2, \ldots, n) = c > 0$$

This is much easier to prove than the lemma.

A key estimate

Suppose that $\zeta_{(W_0)} = 0$ and $(W_0(-k))_{k\geq 0}$ is distributed as a random walk with jump distribution μ started from 0.

Write $\widehat{W}_k := W_k(\zeta_{(W_k)})$ for the terminal point of W_k (the "head of the discrete snake")

Lemma

We have

$$\lim_{n\to\infty} (\log n)\, P(\widehat{W}_k \neq \widehat{W}_0 \text{ for all } k = 1, 2, \ldots, n) = 4\pi^2\sigma^4.$$

Remark. Analogous result for a (centered, finite variance) random walk S on $\mathbb{Z}^2$ started from 0,

$$\lim_{n\to\infty} (\log n)\, P(S_k \neq 0 \text{ for all } k = 1, 2, \ldots, n) = c > 0$$

This is much easier to prove than the lemma.

Application of the key estimate 1

Set $R_n = \#\{\widehat{W}_0, \widehat{W}_1, \ldots, \widehat{W}_n\}$.
Then,

$$E[R_n] = E\left[\sum_{j=0}^{n} \mathbf{1}_{\{\widehat{W}_k \neq \widehat{W}_j, \text{ for all } k=j+1,\ldots,n\}}\right]$$

$$= \sum_{j=0}^{n} P\left[\widehat{W}_k \neq \widehat{W}_j, \text{ for all } k = j+1, \ldots, n\right]$$

$$= \sum_{j=0}^{n} P\left[\widehat{W}_k \neq \widehat{W}_0, \text{ for all } k = 1, \ldots, n-j-1\right]$$

by stationarity. The lemma now gives

$$\lim_{n\to\infty} \frac{\log n}{n} E[R_n] = 4\pi^2\sigma^4.$$

Application of the key estimate 2

By similar arguments,

$$\lim_{n\to\infty} \left(\frac{\log n}{n}\right)^2 E[(R_n)^2] = (4\pi^2\sigma^4)^2.$$

Suffices to get

$$\frac{\log n}{n}\, R_n \xrightarrow[n\to\infty]{L^2} 4\pi^2\sigma^4.$$

$\longrightarrow$ This gives the desired asymptotics for the range of random walk indexed by the infinite geometric tree.

$\longrightarrow$ Technical work (more difficult than in the supercritical case) is needed to get the asymptotics for a tree with fixed size n.

Ideas of the proof of the main estimate 1

We aim at proving that

$$P(\widehat{W}_k \neq \widehat{W}_0 \text{ for all } k = 1, 2, \ldots, n) \underset{n\to\infty}{\sim} \frac{4\pi^2\sigma^4}{\log n}.$$

Start from the identity

$$\begin{aligned} 1 &= \sum_{k=0}^{n} P(\widehat{W}_k = 0; \widehat{W}_\ell \neq 0, \forall \ell \in \{k+1, \ldots, n\}) \\ &= \sum_{k=0}^{n} E\Big[\mathbf{1}_{\{\widehat{W}_k = 0\}}\, P_{W_k}(\widehat{W}_\ell \neq 0, \forall \ell \in \{1, \ldots, n-k\})\Big] \quad (\textit{Markov}) \\ &= \sum_{k=0}^{n} E\Big[\mathbf{1}_{\{\widehat{W}_k = 0\}}\, P_{W_0}(\widehat{W}_\ell \neq 0, \forall \ell \in \{1, \ldots, n-k\})\Big]. \end{aligned}$$

(symmetry argument: (W_0, W_k) and (W_k, W_0) have the same distribution under $P(\cdot \mid \widehat{W}_k = 0)$)

Ideas of the proof of the main estimate 2

It follows that

$$1 = E\left[E_{W_0}\left[\sum_{k=0}^{n} \mathbf{1}_{\{\widehat{W}_k = 0\}} \right] P_{W_0}(\widehat{W}_\ell \neq 0, \forall \ell \in \{1, \ldots, n-k\}) \right].$$

Direct calculations show that

$$E\left[\sum_{k=0}^{n} \mathbf{1}_{\{\widehat{W}_k = 0\}} \right] \underset{n\to\infty}{\sim} \frac{\log n}{4\pi^2\sigma^4}.$$

Needs to verify that

$$E_{W_0}\left[\sum_{k=0}^{n} \mathbf{1}_{\{\widehat{W}_k = 0\}} \right]$$

is very concentrated near its mean:

$\longrightarrow$ First get a continuous version of this concentration property involving Brownian motion

$\longrightarrow$ Then use a strong invariance principle (Komlós-Major-Tusnády and Zaitsev) to complete the proof.