On a conjecture by Gallai and a question by Erdős

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A (proper) k-coloring of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, \ldots, k\}$ such that $f(u) \neq f(v)$ for each $uv \in E$.

A graph G is k-colorable if it has a k-coloring.
The chromatic number, $\chi(G)$, of a graph G is the smallest k such that G is k-colorable.

A graph G is k-critical if G is not $(k-1)$-colorable, but every proper subgraph of G is $(k-1)$-colorable.

Every k-critical graph has chromatic number k and every k-chromatic graph contains a k-critical subgraph.

Erdős on critical graphs

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Erdős wrote in 1989:
Dirac defined a k-chromatic graph to be vertex critical if the omission of any vertex decreases the chromatic number and edge critical if the removal of any edge decreases the chromatic number. I immediately liked these concepts very much and in fact felt somewhat foolish that I did not think of these natural and obviously fruitful concepts before.

Critical graphs

The only 1-critical graph is K_1, and the only 2-critical graph is K_2. The only 3-critical graphs are the odd cycles.

For every $k \geq 4$ and every $n \geq k+2$, there exists a k-critical n-vertex graph.

Every k-critical graph is 2-connected and $(k-1)$-edge-connected.

$f_k(n)$

$f_k(n)$ — the minimum number of edges in a k-critical graph with n vertices.

Since $\delta(G) \geq k - 1$ for every k-critical graph G,

$$f_k(n) \geq \frac{k-1}{2} n \qquad (1)$$

for all $n \geq k$, $n \neq k + 1$.

Brooks' Theorem implies that for $k \geq 4$ and $n \geq k + 2$, the inequality in (1) is strict.

Dirac's bound

Dirac in 1957 asked to determine $f_k(n)$ and proved that for $k \geq 4$ and $n \geq k+2$,

$$f_k(n) \geq \frac{k-1}{2}n + \frac{k-3}{2}. \tag{2}$$

The result is tight for $n = 2k-1$ and yields

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Stiebitz and A.K. improved (2) to

$$f_k(n) \geq \frac{k-1}{2}n + k - 3 \tag{3}$$

when $n \neq 2k-1, k$.

This yields $f_k(2k) = k^2 - 3$ and $f_k(3k-2) = \frac{3k(k-1)}{2} - 2$.

Gallai's results and conjecture

Theorem 1 [Gallai, 1963] If $k \geq 4$ and $k+2 \leq n \leq 2k-1$, then

$$f_k(n) = \frac{1}{2}((k-1)n + (n-k)(2k-n)) - 1.$$

Theorem 2 [Gallai, 1963] For all $k \geq 4$ and $n \geq k+2$,

$$f_k(n) \geq \frac{k-1}{2}n + \frac{k-3}{2(k^2-3)}n. \quad (4)$$

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Conjecture 1 [Gallai, 1963] If $k \geq 4$ and $n = 1 \pmod{k-1}$, then $f_k(n) = \frac{(k+1)(k-2)n-k(k-3)}{2(k-1)}$.

Hajós' Construction

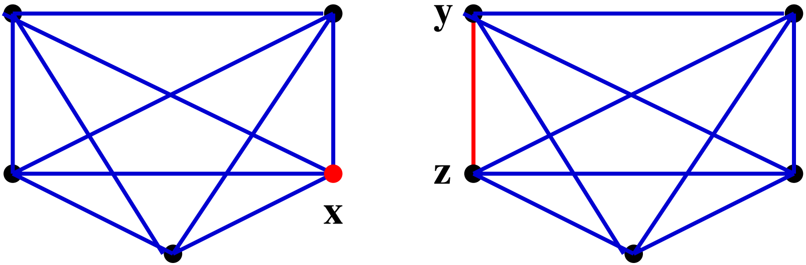

Figure: Choose a vertex x in one k-critical graph and an edge yz in the other.

Hajós' Construction-2

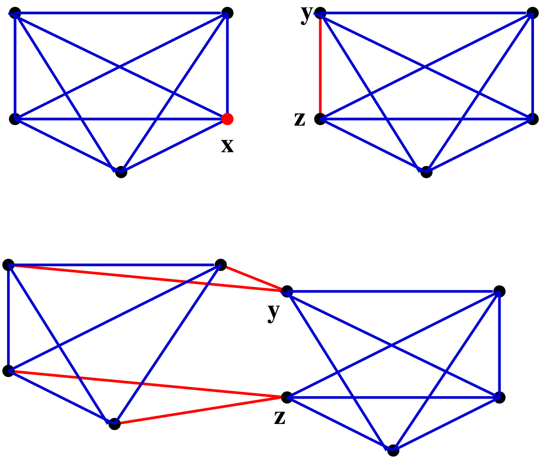

Figure: Delete yz, split x and glue the two pieces of x to y and z. Call the new graph $H(G_1, G_2)$.

Ore's Conjecture

Ore observed that Hajós' construction implies

$$f_k(n+k-1) \leq f_k(n) + (k-1)\left(\frac{k}{2} - \frac{1}{k-1}\right), \tag{5}$$

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which yields that $\phi_k := \lim_{n\to\infty} \frac{f_k(n)}{n}$ exists and satisfies

$$\phi_k \leq \frac{k}{2} - \frac{1}{k-1}. \tag{6}$$

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Ore (1967) conjectured that for every $n \ge k+2$, in (4) equality holds.

Other results

Krivelevich (1997), Stiebitz and A.K. (2003), Farzad and Molloy (2009).

The problem of finding $f_k(n)$ is Problem 5.3 in the monograph of Jensen and Toft and Problem 12 in their list of 25 pretty graph colouring problems. It is a half of Problem P1 in the Handbook of Graph Theory.

Theorem 3 [Yancey and A.K.] If $k \geq 4$ and G is k-critical, then $|E(G)| \geq \left\lceil \frac{(k+1)(k-2)|V(G)|-k(k-3)}{2(k-1)} \right\rceil$. In other words, if $k \geq 4$ and $n \geq k$, $n \neq k+1$, then

$$f_k(n) \geq F(k,n) := \left\lceil \frac{(k+1)(k-2)n - k(k-3)}{2(k-1)} \right\rceil . \qquad (7)$$

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Corollary 1: For every $k \geq 4$ and $n \geq k+2$,

$$0 \leq f_k(n) - F(k,n) \leq \frac{k(k-1)}{8} - 1.$$

In particular, $\phi_k = \frac{k}{2} - \frac{1}{k-1}$ and $f_4(n) = F(4,n)$ for every $n \geq 6$.

Extremal graphs

A k-extremal graph is a k-critical graph G such that $|E(G)| = \frac{(k+1)(k-2)|V(G)| - k(k-3)}{2(k-1)}$.

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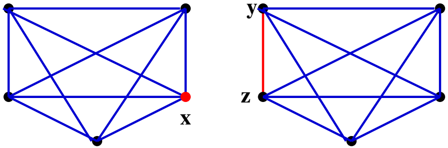

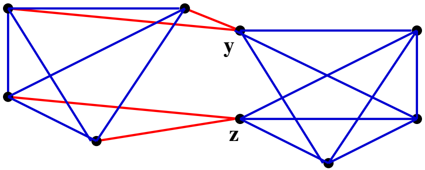

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A *k*-extremal graph is a k-critical graph G such that $|E(G)| = \frac{(k+1)(k-2)|V(G)|-k(k-3)}{2(k-1)}$.

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A graph is k-Ore if it is obtained by a sequence of Hajos' constructions from a set of copies of K_k.

By above, every k-Ore graph is k-extremal.

So, for every $k \geq 4$, there are infinitely many k-extremal graphs.

Theorem 4 [Yancey and A.K.] Let $k \geq 4$ and G be a k-critical graph. Then G is k-extremal if and only if it is a k-Ore graph. Moreover, if G is not a k-Ore graph, then $|E(G)| \geq \frac{(k^2-k-2)|V(G)|-y_k}{2(k-1)}$, where $y_k = \max\{2k-6, k^2-5k+2\}$.

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This gives a slightly better approximation for $f_k(n)$ and adds new cases where we now know the exact values of $f_k(n)$. In particular, we know $f_5(n)$ for every $n \geq 7$.

The value of y_k in Theorem 4 is best possible in the sense that, as observed by Bjarne Toft, for every $k \geq 4$, there exists an infinite family of 3-connected graphs with $|E(G)| = \frac{(k^2-k-2)|V(G)|-y_k}{2(k-1)}$.

Planar graphs

Grötzsch's Theorem Every planar triangle-free graph is 3-colorable.

Theorem 5 [Jensen and Thomassen] If a graph G is obtained from a planar triangle-free graph H by adding a vertex of degree at most 3, then G is 3-colorable.

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There are infinitely many 4-critical graphs obtained from a planar triangle-free graph by adding a vertex of degree 5.

A sharpening of Theorem 5

Theorem 6 [Borodin, A.K., Lidický and Yancey] If a graph G is obtained from a planar triangle-free graph H by adding a vertex of degree at most 4, then G is 3-colorable.

Proof: Let G be a smallest counter-example. Then G is 4-critical and so 2-connected. Let v be the vertex added to a planar triangle-free H. Suppose G has n vertices and e edges. Suppose H has f faces, n' vertices and e' edges.
Clearly, $n' = n - 1$ and $e' \geq e - 4$.

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$$e \geq (5n - 2)/3. \tag{8}$$

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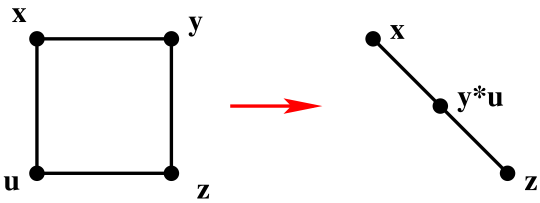
Folklore observation: H has no 4-faces.
Then $2e' \geq 5f$. Plug this into Euler's Formula $n' - e' + f = 2$:

$$n' - e' + \frac{2e'}{5} \geq 2, \quad \text{i.e.} \quad (n-1) - 2 \geq \frac{3(e-4)}{5}.$$

So, $5n - 3 \geq 3e$, a contradiction to (6).

Proof of the folklore observation



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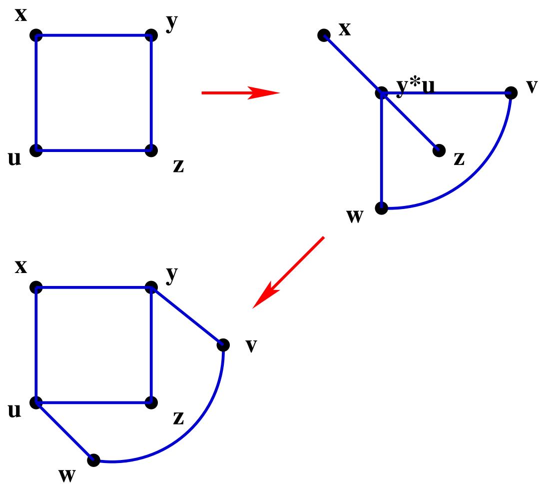

Grünbaum–Aksenov Theorem

Theorem 7 [Grünbaum–Aksenov] Every planar graph with at most three triangles is 3-colorable.

K_4 shows that "three" in Theorem 7 cannot be replaced by "four".

But maybe there are not many plane 4-critical graphs with exactly four triangles ($4, 4$-*graphs*, for short)?

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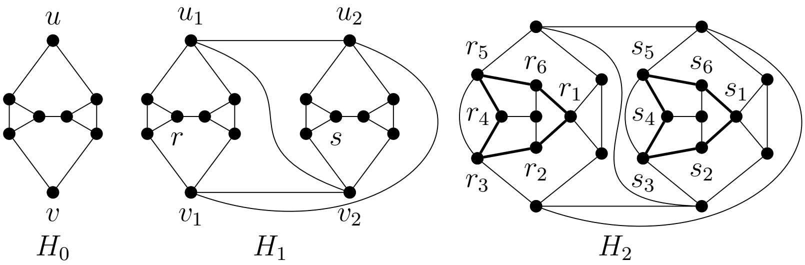
But maybe there are not many plane 4-critical graphs with exactly four triangles ($4, 4$-*graphs*, for short)?

It turned out that there are many.
Havel in 1969 presented a $4, 4$-graph H_1 in which the four triangles had no common vertices.

Havel used the *quasi-edge* $H_0 = H_0(u, v)$ (on the left):



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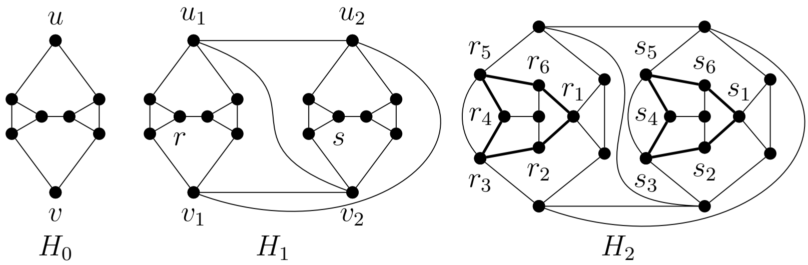

Figure: A quasi-edge H_0 and $4, 4$-graphs H_1 and H_2.

Sachs in 1972 asked whether it is true that in every non-3-colorable planar graph G with exactly four triangles and no separating triangles, these triangles can be partitioned into two pairs so that in each pair the distance between the triangles is less than two.

Aksenov and Mel'nikov in 1978 answered to the question in the negative by constructing a 4, 4-graph H_2 (on the right):

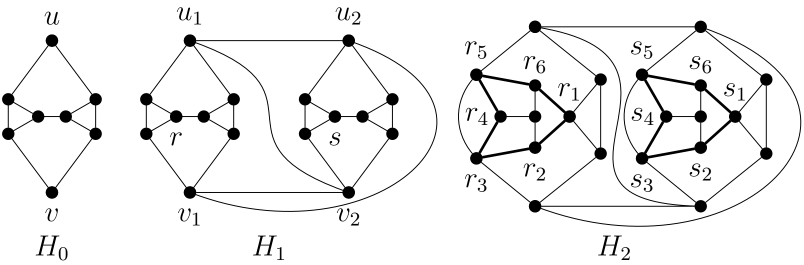

Figure: A quasi-edge H_0 and 4, 4-graphs H_1 and H_2.

Moreover, they constructed two infinite series of 4, 4-graphs.

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Thomas and Walls constructed an infinite family $\mathcal{TW}$ of $4,4$-graphs that have no 4-faces (we will call $4,4$-graphs with no 4-faces *$4,4,4f$-graphs*):

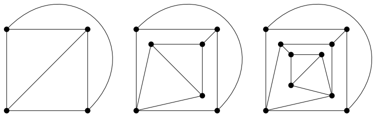

Figure: Smallest Thomas-Walls graphs.

Note that H_1 is a $4,4,4f$-graph but is not in $\mathcal{TW}$. Graph H_2 is not a $4,4,4f$-graph.

4-Ore graphs

First we describe the family $\mathcal{P}_{4,4,4}$ of all $4,4,4f$-graphs.

Recall that a graph is 4-Ore if it is obtained from a set of copies of K_4 by a sequence of the above Hajos' constructions. Every 4-Ore graph is 4-critical.

By Euler's Formula, every $4,4,4f$-graph is a 4-Ore graph.

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On the other hand,

Theorem 8 [Borodin, Dvořák, A. K., Lidický, and Yancey] Every 4-Ore graph has at least four triangles. Moreover, a 4-Ore graph G has exactly four triangles if and only if G is a $4,4,4f$-graph.

A characterization of $4, 4, 4f$-graphs

Theorem 9 [B-D-K-L-Y] Every $4, 4, 4f$-graph is either in $\mathcal{TW}$ or is obtained from a graph in $\mathcal{TW}$ by replacing one or both diamond edges by the Havel's quasi-edge H_0.

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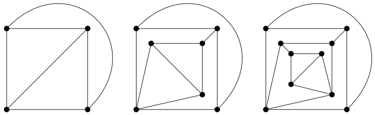
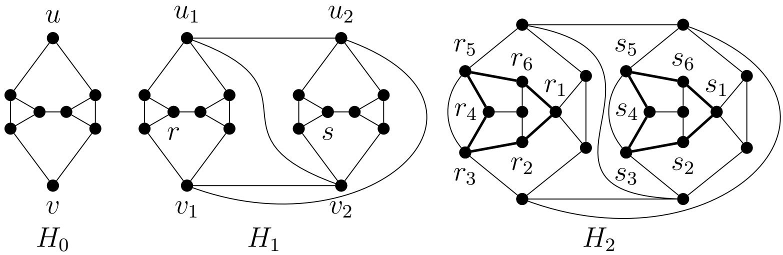
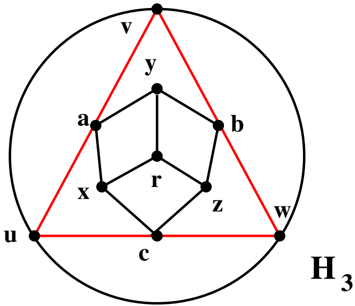




Example: A $4,4$-graph with no 5-faces

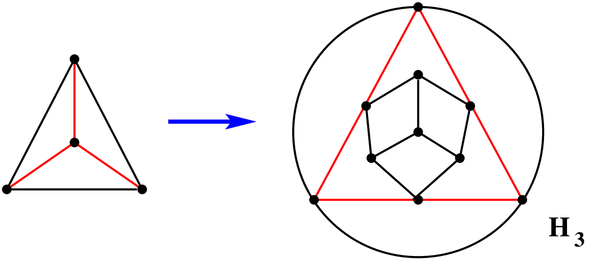


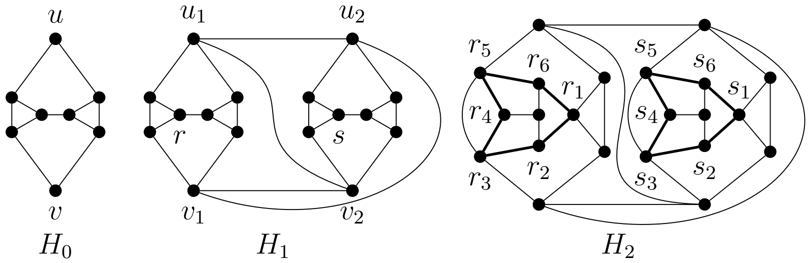
Patches

A *patch* P is a subgraph of a plane graph such that
a) the boundary of P is a 6-cycle $C_P = (x, z', y, x', z, y')$, all vertices on C_P and inside C_P are in P;
b) vertices x', y', z' have no neighbors outside of P,
c) all faces inside C_P are 4-faces.

Observation: If G is a $4, 4$-graph and a vertex $x \in V(G)$ has exactly 3 neighbors, x, y and z, then the graph G_v obtained from $G - v$ by inserting a patch P with $C_P = (x, z', y, x', z, y')$ where x, y, z are old and x', y', z' are new vertices is again a $4, 4$-graph.

Examples





Main result

Theorem 10 [B-D-K-L-Y] A plane 4-critical graph has exactly four 3-cycles if and only if it is obtained from a $4, 4, 4f$-graph by replacing several (maybe zero) non-adjacent 3-vertices with patches.

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This fully answers the question of Erdős from 1990.

So, Sachs had right intuition in 1972: his question has positive answer if we replace "less than two" with "at most two".

Aksenov and Mel'nikov in 1979 conjectured, in particular, that H_1 is the unique smallest $4, 4$-graph with the minimum distance 1 between triangles and H_2 is the unique smallest $4, 4$-graph with the minimum distance 2 between triangles. Our description confirms this.