FROM INTERSECTION

TO CAPACITY

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Erdős-Ko-Rado (1961)

A set family $\mathcal{F} \subseteq 2^{[n]}$ is intersecting if

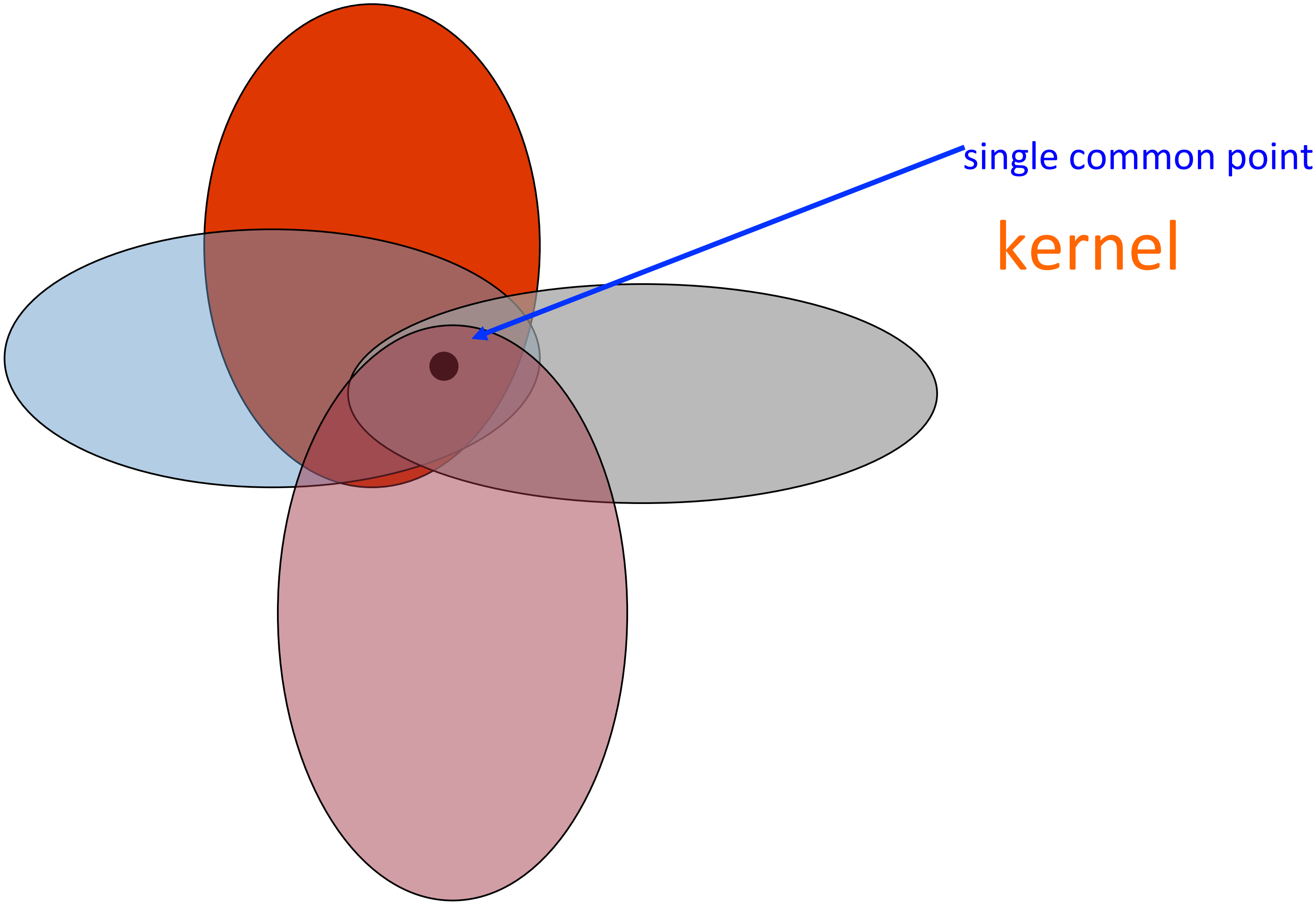
$A \in \mathcal{F},\ B \in \mathcal{F} \Rightarrow A \cap B \neq \emptyset$

$M(k,n)=\max\{|\mathcal{C}|;\ \mathcal{C} \subseteq \binom{[n]}{k},\ \mathcal{C}\text{ intersecting}\}$

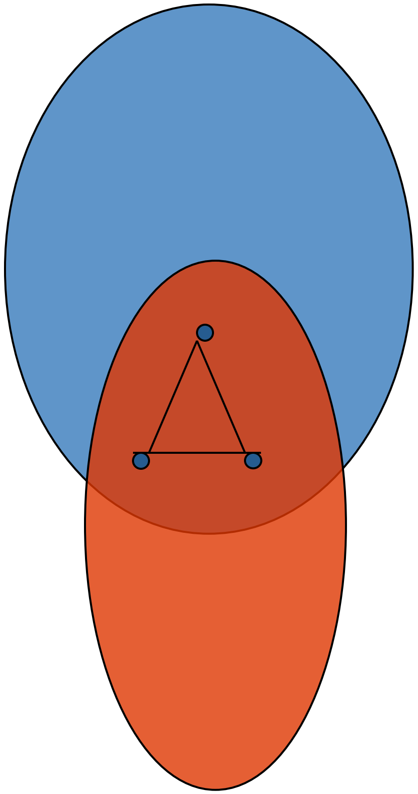
Theorem EKR

$$M(k,n)= \begin{cases} \binom{n}{k} & \text{if } k> n/2 \\ \binom{n-1}{k-1} & \text{else} \end{cases}$$

EKR: Unique optimal construction



Intersection problems



Simonovits-Sós (1976)

How many graphs with vertex set [n] can be found with their pairwise intersection containing a triangle?

Conjecture Simonovits - Sós:

Unique optimum:
All graphs w. a fixed triangle

Proved: David Ellis, Ehud Friedgut, Yuval Filmus (2011)

Dual formulation: union-closed families

$\mathcal{F}$ family of all graphs G on [n] with

α(G) ≥ 3

$C \subseteq \mathcal{F}$ is union-closed if the pairwise union

of its elements is in $\mathcal{F}$

max $|C|$ achieved by all graphs containing a fixed induced stable set of 3 vertices

Kernel structure

A family of strings (e. g. graphs described by the characteristic vectors of their edge set) is a kernel structure if

for a set of coordinates they all have the same projection.

Extremal union-closed families often have a kernel structure.

Two-family generalization

$\mathcal{F}$ and $\mathcal{G}$ not necessarily distinct graph families on [n]

Let M($\mathcal{F}$, $\mathcal{G}$) be the maximum cardinality of a $C \subseteq \mathcal{F}$

such that for any two distinct A$\in C$, B$\in C$

$$A \cup B \in \mathcal{G}$$

If $\mathcal{F}=\mathcal{G}$, then M($\mathcal{F}$, $\mathcal{F}$) is the largest cardinality of a union-closed graph family in $\mathcal{F}$.

However, $\mathcal{F}$ and $\mathcal{G}$ are arbitrary and can be very different.

Let $\mathcal{F} = \mathcal{F}(n, k)$ be the family of all graphs on [n] with

at least k connected components.

$\mathcal{G}$ the connected graphs on [n]

The graphs in $\mathcal{F}$ are very disconnected and thus very different from the connected graphs in $\mathcal{G}$..

What is the value of M($\mathcal{F}$, $\mathcal{G}$) ?

Theorem (Cohen, Fachini, K., (2013+))

$$\lim_{n\to\infty} 1/n \log_2 M(\mathcal{F}(n,k), \mathcal{G}) = h(1/k)$$

____

where $h(t) = -\log_2 t - \log_2(1-t)$ is the binary entropy function.

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Sketch of proof:

$\leq$

Claim:

$M(\mathcal{F}(n,k), \mathcal{G}) \leq \sum_{i=1}^{n/k} \binom{n}{i} \leq$

$\leq \exp_2 [nh(1/k)]$

W. l. o. g. suppose optimal

$\mathcal{C} \subseteq \mathcal{F}$ (n,k) consists of complements of complete k-partite graphs.

Every component uniquely belongs to its graph.
Thus each graph has a clique of size $\leq n/k$ of its own.

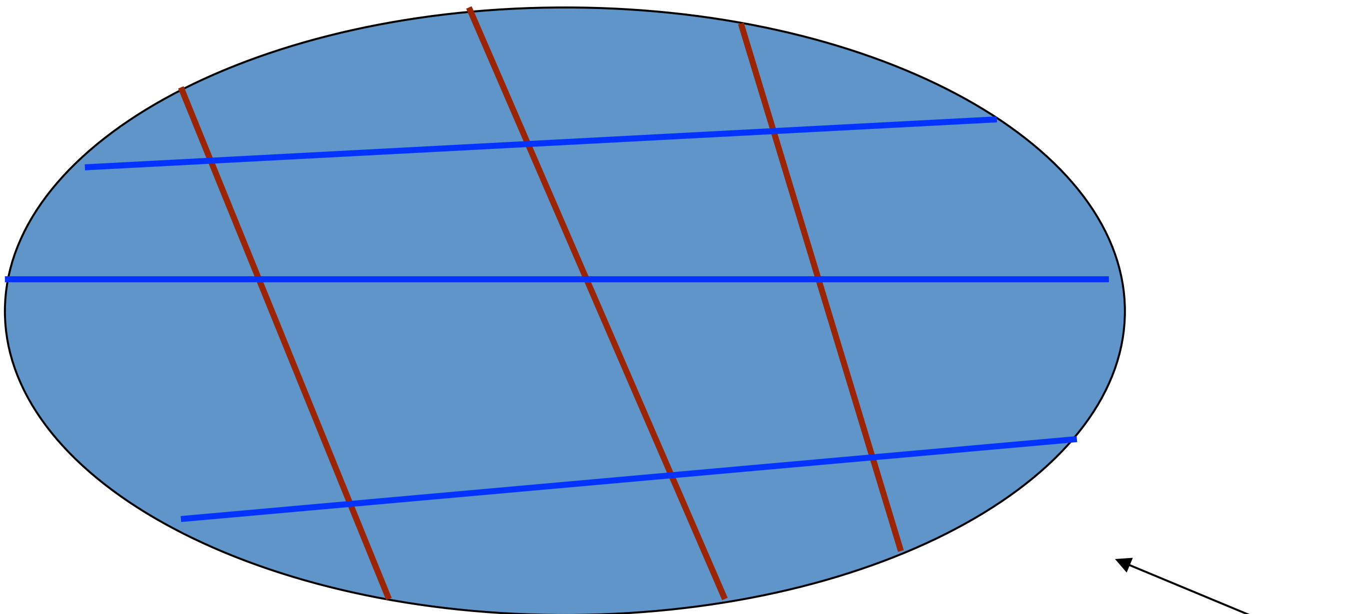
$\Downarrow$

$$|\mathcal{C}| \leq \exp_2 [nh(1/k)]$$

≥

Example: two qualitatively independent k-partitions:

The union graph is connected.



k=4

Less is needed.

$\geq$

$\mathbf{x} \in [k]^n$, $\mathbf{y} \in [k]^n$

Characteristic vectors of two partitions representing graphs

Sufficient condition for the union to be connected:

.

$\mathbf{x}$ ● ● ● ● ● ● ●…………x_i…………● ● 1 2 ……..k

$\mathbf{y}$ ● ● ● ● ● ● ●…………y_i…………● ● 1 2 ……..k

For all $A \subseteq [k]$ $\exists$ i $x_i \in A$ $y_i \notin A$

$\geq$

$\mathbf{x} \in [k]^n$, $\mathbf{y} \in [k]^n$

Characteristic vectors of two partitions representing graphs

Sufficient condition for the union to be connected:

$a \in A$

x ● ● ● ● ● ● ●............x_i............● ● 1 2 ...b.....k

$b \notin A$

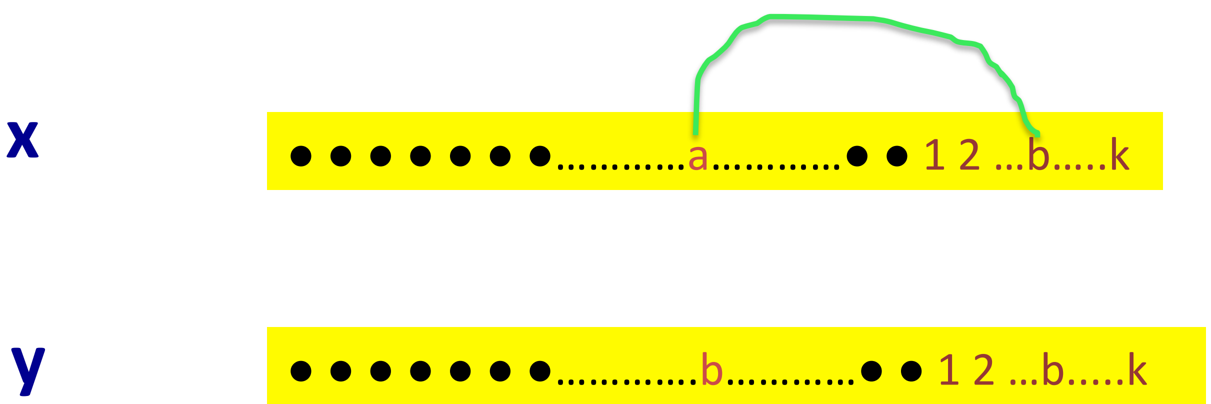
y ● ● ● ● ● ● ●............y_i............● ● 1 2 ......b...k

For all $A \subseteq [k]$ $\exists$ i $x_i \in A$ $y_i \notin A$

$\geq$

$\mathbf{x} \in [k]^n$, $\mathbf{y} \in [k]^n$

x_i of class a and the class b of x connected in y as vertices of its same class b



A maximum cardinality set of sequences with this property can be constructed using the

capacity result for graph families of Gargano-K-Vaccaro

$\mathfrak{B}$ the family of all bipartite complete graphs on [k]

$\mathcal{C}_n \subseteq [k]^{n-k}$

a clique in the (n-k)'th power of all the graphs in $\mathfrak{B}$

Adding the postfix 12…k to the strings in $\mathcal{C}$ we get the desired construction of graphs.

By theorem Gargano-K- Vaccaro

$\exists$ $\mathcal{C}_n$, n=1,2,... with

$$n^{-1} \log_2 |\mathcal{C}_n| \gtrsim \min_{\{A \subseteq [k]\}} h(|A|/k)$$

$$= h(1/k)$$

____

Examples of other pairs of graph families

$\mathcal{F}_n$ all the Hamilton paths in K_n

$\mathcal{G}$ graphs on [n] containing K_4

Theorem (K-Messuti-Simonyi (2012))

$$\exp_2 (\lfloor n/4 \rfloor) \leq M(\mathcal{F}_n, \mathcal{G}) \leq (n+1)^2 (3/2)^{(n-1)}$$

Open problem

$\mathcal{F}_n$ all the Hamilton paths in K_n

$\mathcal{G}$ graphs on [n] containing K_3

$M(\mathcal{F}_n, \mathcal{G}) = ?$

many more…

Examples of other pairs of graph families

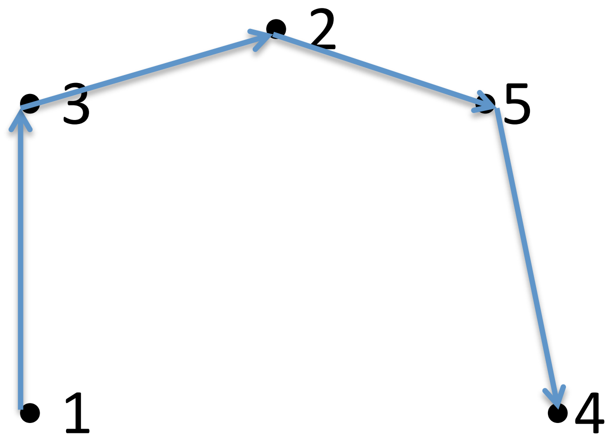
$\mathcal{F}_n$ all the graphs of constant degree 2 on [n]

$\mathcal{G}$ graphs of maximum degree ≥ 4

Theorem (K-Muzi (2013+))

$$n![(n/3)!]^{-1}\, 6^{-n/3} \leq M(\mathcal{F}_n, \mathcal{G}) \leq n![(n/3)!]^{-1}\, e^{\sqrt{n}}$$

More relations to graph capacity problems



1,3,2,5,4

n=5

Permutation of [n] ↔ Hamilton path in K_n

Q: How many Hamilton paths in K_n such that any two differ in some specific way?

Permutation language:

How many permutations of [n] can we have so that

any two differ in some specific way?

$C \subseteq S_n \quad \pi \in C, \quad \rho \in C, \quad \pi \neq \rho$

$\exists\ i \in [n] \qquad | \pi(i)-\rho(i) | = 1$

$T(n) = \max\ |C|$

(K-Malvenuto, 2006)

Current bounds

$$2^{0.8604n} \leq T(n) \leq \binom{n}{\lfloor n/2 \rfloor}$$

Brightwell
Cohen
Fachini
Fairthorne
K
Simonyi
Tóth
(2011)

Conjectured tight, K-Malvenuto (2006)

Generalization: permutation capacity of infinite graphs

G infinite graph, $V(G)=\mathbf{N}$ (natural numbers)

$C \subseteq S_n$ $\pi \in C$, $\rho \in C$, $\pi \neq \rho$

$\exists\ i \in [n]$ $\{\pi(i), \rho(i)\} \in E(G)$

This generalizes Shannon capacity.

THE END