Expansions in non-integer bases

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Abstract

In 1957 Rényi extended the familiar integer base expansions to non-integer bases.

Erdős, Horváth and Joó started around 1990 to investigate the structure of all, not necessarily greedy expansions.

We present some results in this field, obtained mostly in collaboration with Erdős, Joó, Loreti, Pethő, de Vries and Akiyama.

For simplicity of exposition we consider only expansions in bases $1 < q \leq 2$, using the two-digit alphabet $\{0, 1\}$.

Outline

Expansions

Greedy or β-expansions (Rényi, 1957)

Given two real numbers $1 < q \le 2$ and $x \ge 0$, there exists a lexicographically largest sequence (b_i) of zeroes and ones, satisfying the inequality

$$\frac{b_1}{q} + \frac{b_2}{q^2} + \frac{b_3}{q^3} + \cdots \le x.$$

If $0 \le x \le 1/(q-1)$, then we have equality here, and (b_i) is called the *greedy or β-expansion* of x in base q.

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Examples

- For $q = 2$ we get the usual binary expansions, by preferring finite expansions when possible.
- For $q = G := (1+\sqrt{5})/2 \approx 1.618$ and $x = 1$ we have

$$1 = \frac{1}{q} + \frac{1}{q^2}.$$

Arbitrary expansions (Erdős et al, 1990–1991)

Given two real numbers $1 < q \leq 2$ and $0 \leq x \leq 1/(q-1)$, by an *expansion* of x in base q we mean a sequence $(c_i) \subset \{0, 1\}$ satisfying

$$x = \frac{c_1}{q} + \frac{c_2}{q^2} + \frac{c_3}{q^3} + \cdots .$$

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Examples

- In base $q = 2$ there are at most two expansions.
- In base $q = G := (1+\sqrt{5})/2 \approx 1.618$, $x = 1$ has $\aleph_0$ expansions, for example $1 = q^{-1} + q^{-2} = q^{-1} + q^{-3} + q^{-4} = \cdots$.
- In bases $1 < q < G$, each $0 < x < 1/(q-1)$ has $2^{\aleph_0}$ expansions.
- Define $q \approx 1.802$ by

$$1 = \frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^4} + \frac{1}{q^6} + \cdots .$$

In this base $x = 1$ has no other expansions.

An example of Erdős and Joó, 1992

Given a positive integer N, set

$$(c_i) := 1^9(0^91)^{N-1}(0^41)^\infty$$

and let $1 < q < 2$ be the solution of

$$1 = \frac{c_1}{q} + \frac{c_2}{q^2} + \frac{c_3}{q^3} + \cdots .$$

In this base, $x = 1$ has exactly N distinct expansions.

Generic picture

- In integer bases a number x has generically a unique expansion.
- (Sidorov, 2003) In non-integer bases a number x has generically $2^{\aleph_0}$ expansions.

Spectra and universal expansions

Universal expansions

A sequence $(c_i) \subset \{0, 1\}$ is called *universal* if it contains all possible finite blocks of digits.

Examples

- In base $q = G$, $x = 1$ has no universal expansion.
- (Erdős–K., 1996) If $q > 1$ is close enough to 1, then every $0 < x < 1/(q - 1)$ has a universal expansion.

Open question. Does the last property hold for all $1 < q < G$?

Spectra of bases

For each fixed $1 < q \le 2$ we set

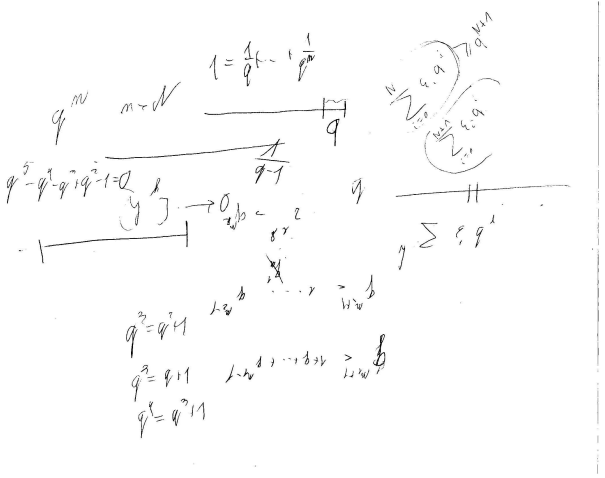
$$Y_q := \{a_0 + a_1 q + \cdots + a_n q^n \ : \ a_0, \ldots, a_n \in \{-1, 0, 1\}, \quad n = 0, 1, \ldots\}.$$

For example, $Y_2 = \mathbb{Z}$ and $Y_{\sqrt{2}} = \mathbb{Z} + \sqrt{2}\mathbb{Z}$.

Proposition

(Erdős–K., 1996) If 0 is an accumulation point of Y_{q^m} for some integer $m > 1$, then every $0 < x < 1/(q-1)$ has a universal expansion in base q.

Spectra and Pisot numbers



Spectra and Pisot numbers

Theorem

(Garsia, 1962) q is a Pisot number $\Longrightarrow Y_q$ *has no accumulation points.*

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Partial converses by Bogmér–Horváth–Sövegjártó, Joó–Schnitzer, Erdős–Joó–K., Erdős–K., Bugeaud, Borwein–Hare, Komatsu, Sidorov–Solomyak, Stankov, Zaimi.

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Theorem

(Akiyama–K., 2011) q is a Pisot number $\Longleftarrow$ Y_q *has no accumulation points.*

Corollary

If $1 < q \le \sqrt[3]{2} \approx 1.26$, *then every* $0 < x < 1/(q-1)$ *has a universal expansion in base q.*

Proof of Akiyama–K.

Immprovement of a method in Erdős–K. (1996):

Lemma

Let $1 < q < 2$ (instead of $1 < q \leq G \approx 1.618$).

- *(Frougny, 1992) If Y_q has no accumulation points, then q is an algebraic integer. Furthermore, if $\sum_{i=0}^{\infty} s_i q^{-i} = 0$ for some sequence $(s_i) \subset \{-1, 0, 1\}$, then we have also $\sum_{i=0}^{\infty} s_i p^{-i} = 0$ for all conjugates q satisfying $|p| > 1$.*
- *If p is a complex number satisfying $|p| > 1$, then there exists a sequence $(s_i) \subset \{-1, 0, 1\}$ such that $\sum_{i=0}^{\infty} s_i q^{-i} = 0$ but $\sum_{i=0}^{\infty} s_i p^{-i} \neq 0$.*
- *Some weaker (but sufficient for our purposes) statements for $|p| = 1$.*

Unique expansions

Univoque bases

Definition

We write $q \in \mathcal{U}$ if $x = 1$ has a unique expansion in base q.

Theorem

(Erdős–Horváth–Joó, 1991, Daróczy–Kátai, 1995)

- *$\mathcal{U}$ is a Lebesgue null set.*
- *$\mathcal{U}$ has the power of continuum.*
- *$\mathcal{U}$ is of the first category.*
- *$\mathcal{U}$ has Hausdorff dimension one.*

Parry type lexicographic characterization

The main tool for the study of unique expansions is the following:

Theorem

(Erdős–Joó–K., 1990) An expansion

$$1 = \frac{c_1}{q} + \frac{c_2}{q^2} + \frac{c_3}{q^3} + \cdots$$

is unique if and only if writing $\overline{c_i} := 1 - c_i$ *we have*

$$\begin{aligned} c_{n+1}c_{n+2}\ldots &< c_1c_2\ldots \quad \textit{whenever} \quad c_n = 0;\\ \overline{c_{n+1}c_{n+2}\ldots} &< c_1c_2\ldots \quad \textit{whenever} \quad c_n = 1. \end{aligned}$$

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Example

$(c_i)=11(01)^\infty$ is a unique expansion.

A constant (K.–Loreti, 1998)

We recall that the Thue–Morse sequence

$$\tau_0, \tau_1, \tau_2, \ldots = 0110\ 1001\ 1001\ 0110\ 1001\ 0110\ 0110\ 1001 \ldots$$

is defined by the recursive formulas

$$\tau_0 := 0 \quad \text{and} \quad \tau_\ell \ldots \tau_{2\ell-1} := \overline{\tau_0 \ldots \tau_{\ell-1}}, \quad \ell = 1, 2, 4, 8, \ldots,$$

where we use the notation $\overline{\tau} := 1 - \tau$.

Let $q' \approx 1.787$ be the positive solution of

$$x = \frac{\tau_1}{q} + \frac{\tau_2}{q^2} + \frac{\tau_3}{q^3} + \cdots .$$

Topology of $\mathcal{U}$

Theorem

- *(K.–Loreti, 1998)* $\mathcal{U}$ *has a smallest element equal to* $q' \approx 1.787$.
- *(K.–Loreti–Pethő, 2003)* q' *is an accumulation point of* $\mathcal{U}$.

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Theorem

(K.–Loreti, 2007)

- $\mathcal{U}$ *is closed from above but not from below.*
- $\overline{\mathcal{U}} \setminus \mathcal{U}$ *is countable and dense in* $\overline{\mathcal{U}}$*.*
- $\overline{\mathcal{U}}$ *is a Cantor set of zero Lebesgue measure.*

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Example

The solution $q \approx 1.839$ of $q^3 = q^2 + q + 1$ belongs to $\overline{\mathcal{U}} \setminus \mathcal{U}$.

Univoque sets in fixed bases

Definition

We write $x \in \mathcal{U}_q$ if x has a unique expansion in base q.

Only the cases $G \leq q < 2$ are interesting.

Theorem

(Glendinning–Sidorov, 2001, de Vries–K., 2009, 2011)

- *If $G \leq q < q'$, then $|\mathcal{U}_q| = \aleph_0$.*
- *If $q' \leq q \leq 2$, then $|\mathcal{U}_q| = 2^{\aleph_0}$.*
- *$\mathcal{U}_q$ is of Hausdorff dimension < 1 if $q < 2$.*
- *The Hausdorff dimension of $\mathcal{U}_q$ converges to 1 as $q \to 2$.*

Topology of $\mathcal{U}_q$

Theorem

(de Vries–K., 2009)

- $\mathcal{U}_q$ *is closed* $\iff q \notin \overline{\mathcal{U}}$.

Topology of $\mathcal{U}_q$

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- $\mathcal{U}_q$ *is closed* $\Longleftrightarrow q \notin \overline{\mathcal{U}}$.
- *If* $q \in \overline{\mathcal{U}}$, *then* $\overline{\mathcal{U}}_q \setminus \mathcal{U}_q$ *is countable and dense in* $\overline{\mathcal{U}}_q$.
- *If* $q \in \overline{\mathcal{U}}$ *and* $q \neq 2$, *then* $\overline{\mathcal{U}}_q$ *is a Cantor set.*

Topology of $\mathcal{U}_q$

Theorem

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The finer topological structure of $\mathcal{U}_q$ depends on whether q belongs to

$$\mathcal{U}, \quad \overline{\mathcal{U}} \setminus \mathcal{U}, \quad \mathcal{V} \setminus \overline{\mathcal{U}} \quad \text{or} \quad (1, \infty) \setminus \mathcal{V},$$

where $\mathcal{V}$ is some particular closed set satisfying $\overline{\mathcal{U}} \subset \mathcal{V} \subset (1, \infty)$ whose smallest element is $G \approx 1.618$.

A two-dimensional univoque set

Definition

We write $(x, q) \in \mathbf{U}$ if x has a unique expansion in base q.

Theorem

(de Vries–K., 2011)

(a) $\mathbf{U}$ *is not closed.* $\overline{\mathbf{U}}$ *is a Cantor set.*

(b) $\mathbf{U}$ *and* $\overline{\mathbf{U}}$ *are two-dimensional Lebesgue null sets.*

(c) $\mathbf{U}$ *and* $\overline{\mathbf{U}}$ *have Hausdorff dimension two.*

Figure: Budapest, July 26, 1996

References

The following two papers constitute the starting point of the research covered here:

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The following overview covers our subject, except the ulterior results obtained in collaboration with S. Akiyama:

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