The regularity method and Ramsey theory

Y. Kohayakawa (São Paulo)

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Budapest

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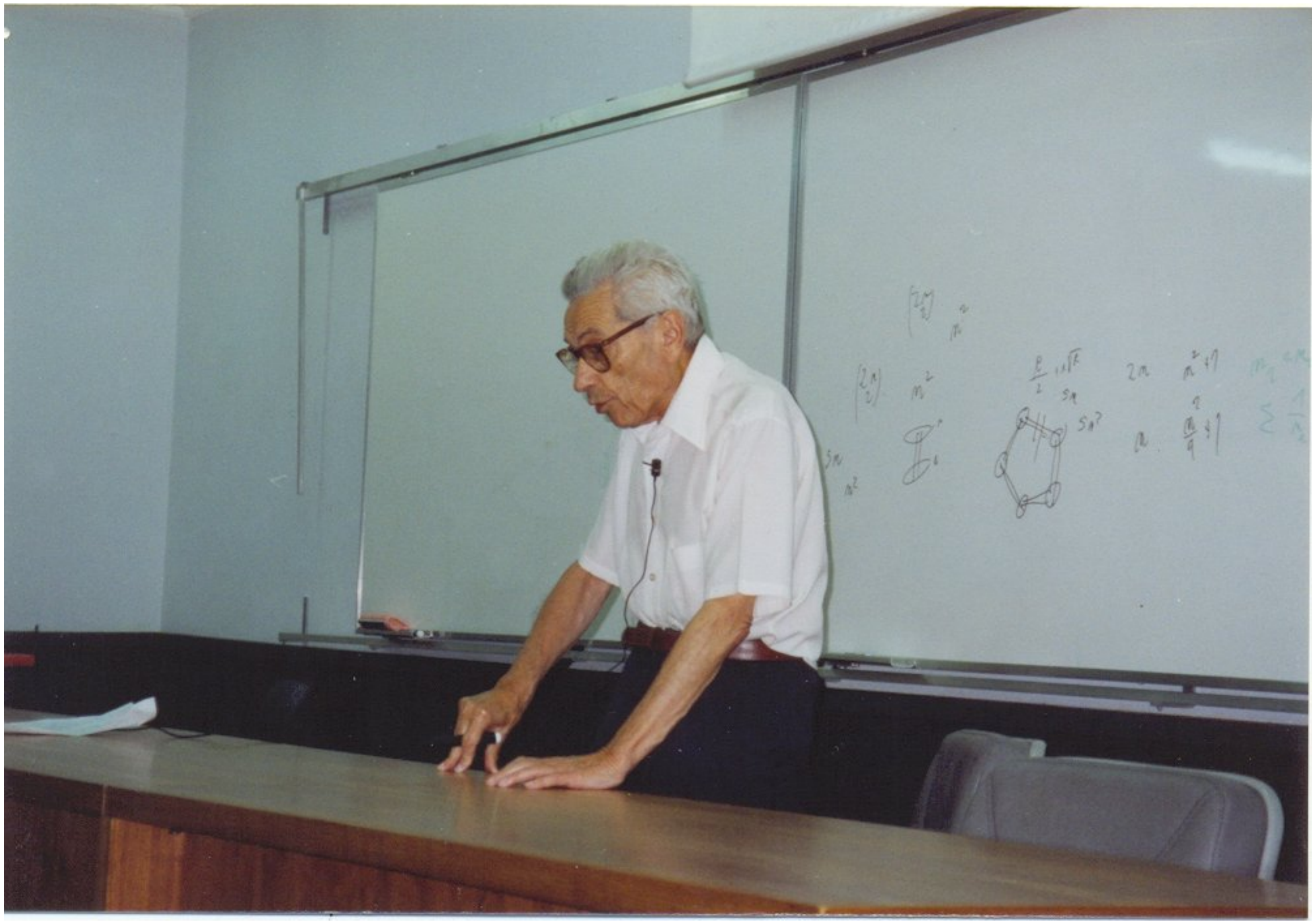

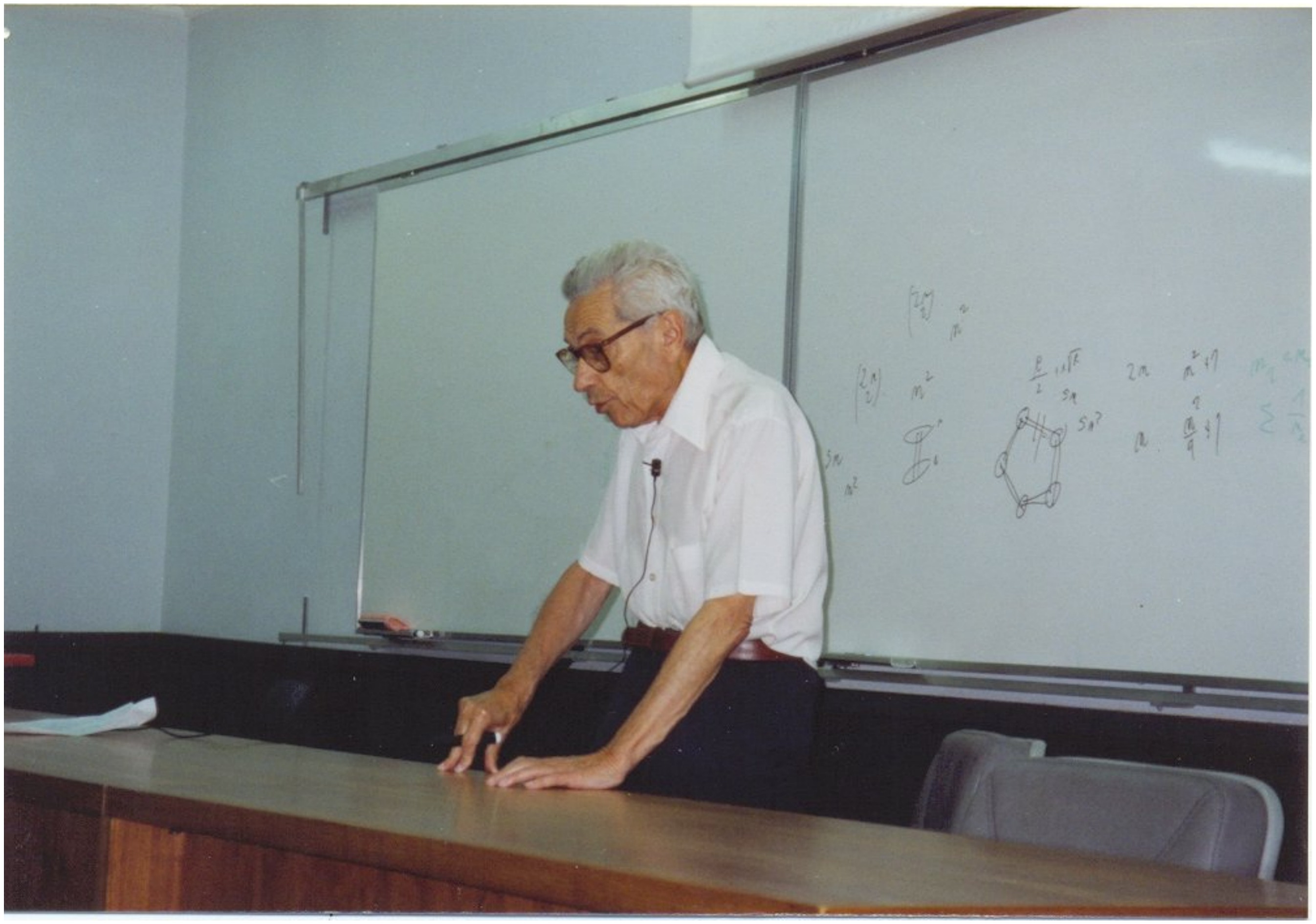

Paul Erdős at the U. of São Paulo, November 1994

RESENHAS

do Instituto de Matemática e Estatística

da Universidade de São Paulo

Special Number Dedicated to Professor Paul Erdős

Resenhas IME-USP, Volume 2, Number 2, 1995

This number of *Resenhas IME-USP* is based on the meeting **Semana de Combinatória** (Instituto de Matemática e Estatística, USP, 9-11 November 1994), which was held in honour of Professor Paul Erdős, on the occasion of his visit to our institute.

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SOME OF MY FAVOURITE PROBLEMS IN NUMBER THEORY, COMBINATORICS, AND GEOMETRY

PAUL ERDŐS

To the memory of my old friend Professor George Svéd.
I heard of his untimely death while writing this paper.

INTRODUCTION

I wrote many papers on unsolved problems and I cannot avoid repetition, but I hope to include at least some problems which have not yet been published. I will start with some number theory.

I. NUMBER THEORY

1. Let $1 \le a_1 < a_2 < \cdots < a_k \le n$ be a sequence of integers for which all the subset sums $\sum_{i=1}^{k} \varepsilon_i a_i$ ($\varepsilon_i = 0$ or 1) are distinct. The powers of 2 have of course of some interest.

2. Covering congruences. This is perhaps my favourite problem. It is really surprising that it has not been asked before. A system of congruences

$$a_i \pmod{n_i}, \qquad n_1 < n_2 < \cdots < n_k \tag{3}$$

is called a *covering system* if every integer satisfies at least one of the congruences in (3). The simplest covering system is 0 (mod 2), 0 (mod 3), 1 (mod 4), 5 (mod 6), 7 (mod 12). The main problem is: Is it true that for every c one can find a covering system all whose moduli are larger than c? I offer 1000 dollars for a proof or disproof.

3. Perhaps it is of some interest to relate the story of how I came to the problem of covering congruences. In 1934 Romanoff **[57]** proved that the lower density of the integers of the form $2^k + p$ (p prime) is positive. This was surprising since the number of sums $2^k + p \le x$ is cx. Romanoff in a letter in 1934 asked me if there were infinitely many odd numbers not of the form $2^k + p$. Using covering congruences I proved in **[27]** that there is an arithmetic progression of odd numbers no term

of which is of the form $2^k + p$. Independently Van der Corput also proved that there are infinitely many odd numbers not of the form $2^k + p$. Crocker **[16]** proved

...

24. Erdős, P., *A generalization of a theorem of Besicovitch*, J. London Math. Soc. **11** (1936), 92–98.
25. ———, *Integral distances*, Bull. Amer. Math. Soc. **51** (1945), 996.
26. ———, *On sets of distances of n points*, Amer. Math. Monthly **53** (1946), 248–250.
27. ———, *On integers of the form $2^k + p$ and some related problems*, Summa Brasiliensis Math. **II** (1950), 113–123.

Overview

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- ▷ **Regularity method**

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- ▷ An application in Ramsey theory: **by Julia Böttcher, Anusch Taraz and Andreas Würfl**

Maximum degree

Definition 1 (Maximum degree)**.** *We denote the maximum degree of a graph* $G = (V, E)$ *by* $\Delta(G)$*, that is,* $\Delta(G) = \max_{v \in V} \deg(v)$*.*

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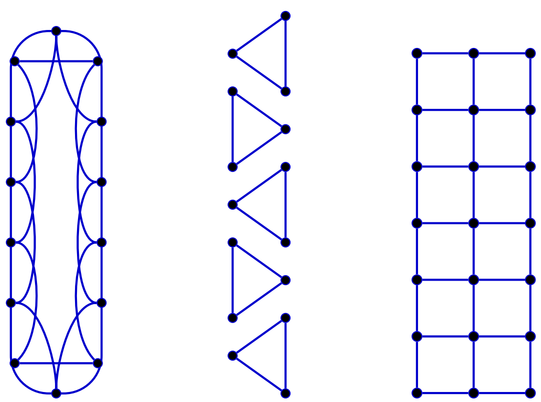
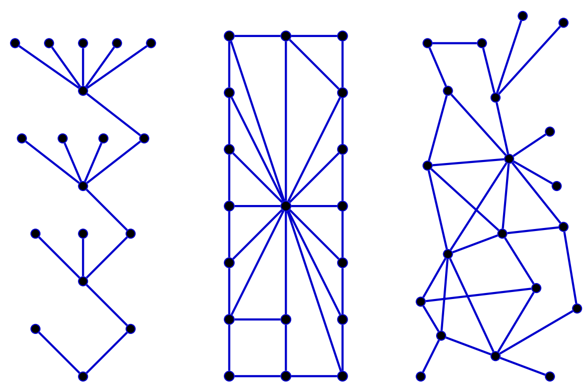
Bounded maximum degree:

- (powers of) cycles
- F-factors
- grids



Unbounded maximum degree:

- trees
- planar graphs
- random graphs



Arrangeability

Arrangeability

Definition 2 (a-arrangeable; Chen and Schelp '93)**.** *A graph* $G = (V, E)$ *is called* a*-arrangeable if there exists an ordering* $x_1 \prec \cdots \prec x_n$ *of* V *with* $|N_L(N_R(x_i))| \leq a$ *for all* $i = 1, \ldots, n$*.*

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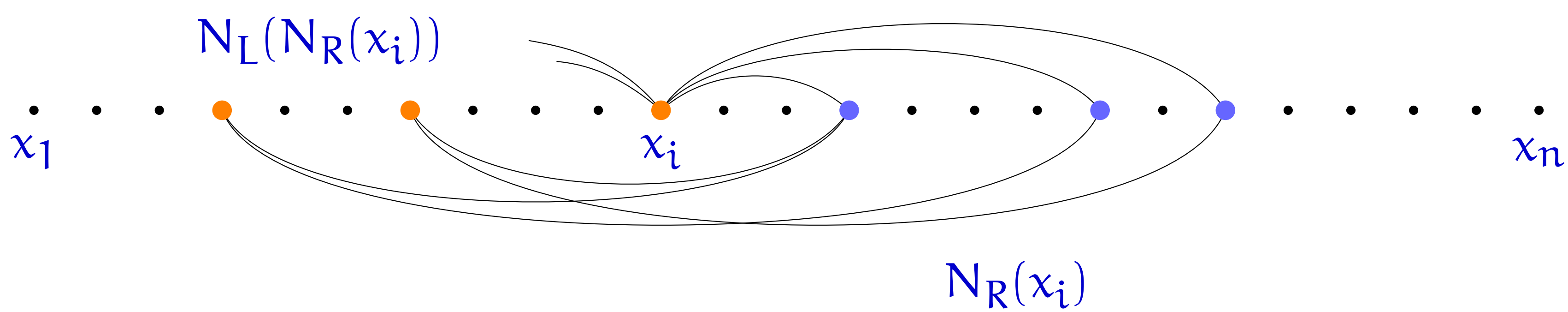

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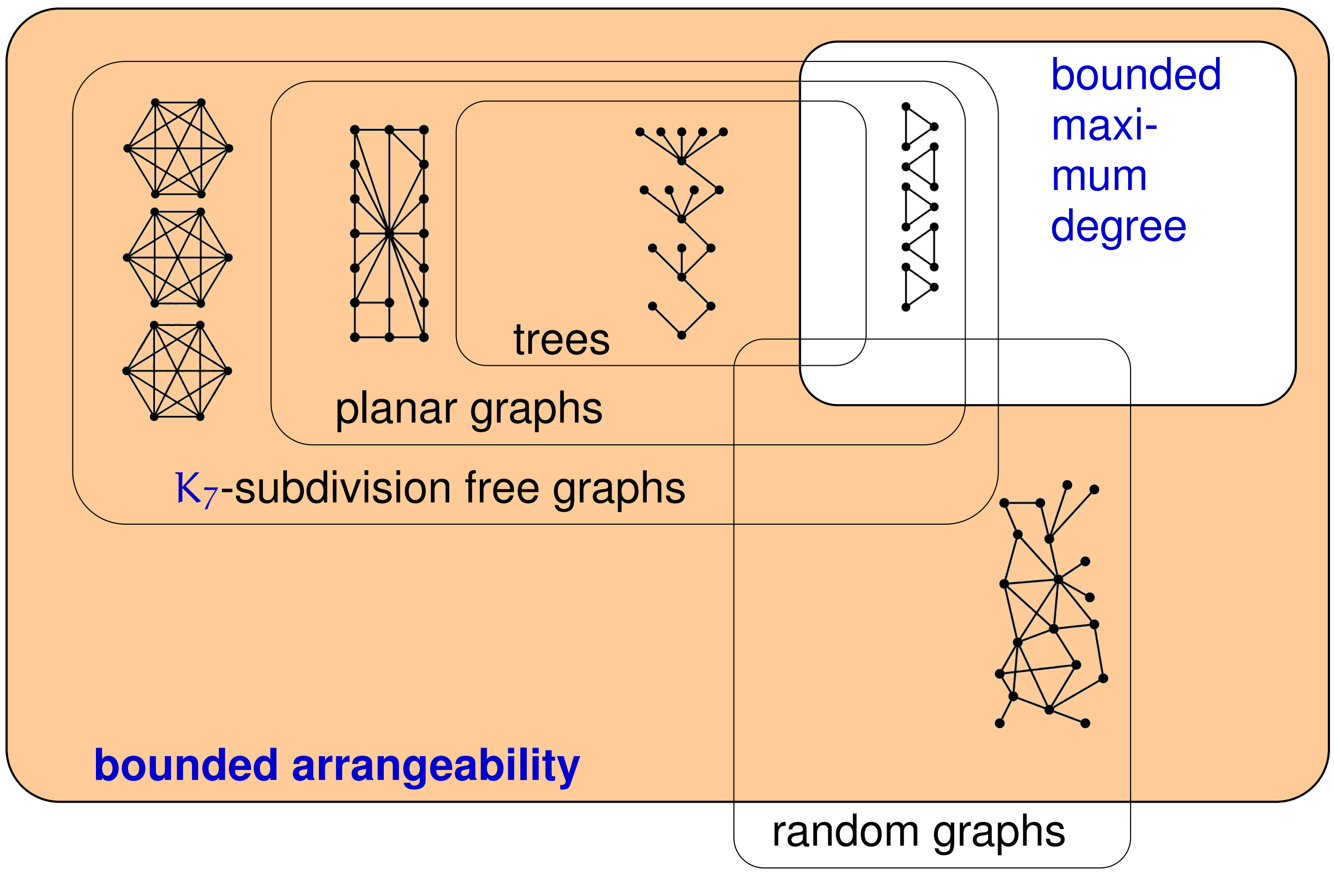
- random graphs on n vertices with dn edges are almost surely $256d^2$-arrangeable [Fox & Sudakov '09]

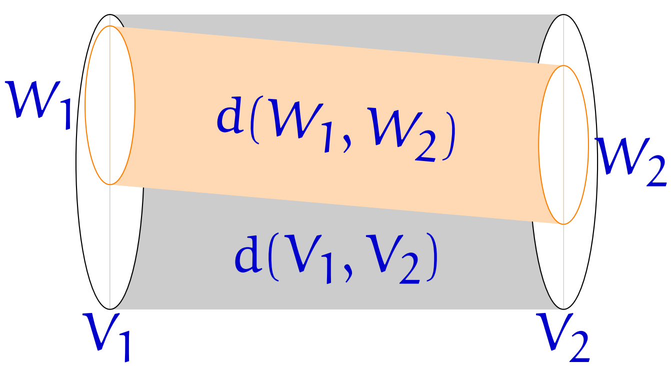
Bounded maximum degree vs bounded arrangeability



The regularity method

Definition 3 ((ε, δ)-super-regular)**.** *Suppose* ε *and* $\delta > 0$*. The graph* $G = (V_1 \cup V_2, E)$ *with* $|V_1| = |V_2| = n$ *is an* (ε, δ)*-super-regular pair if*

- $|d(W_1, W_2) - d(V_1, V_2)| \leq \varepsilon$ *for all* $W_1 \subseteq V_1$, $W_2 \subseteq V_2$ *with* $|W_1|, |W_2| \geq \varepsilon n$*,*
- $\deg(v) \geq \delta n$ *for all* $v \in V_1 \cup V_2$*.*



$$d(W_1, W_2) = \frac{e(W_1, W_2)}{|W_1||W_2|}$$

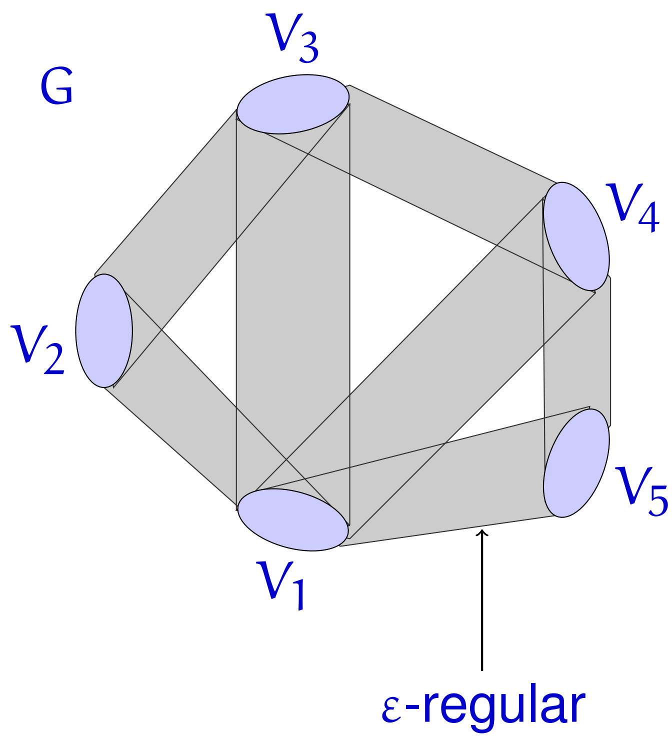
- "regularity": densities equally distributed
- "super-": high minimum degree

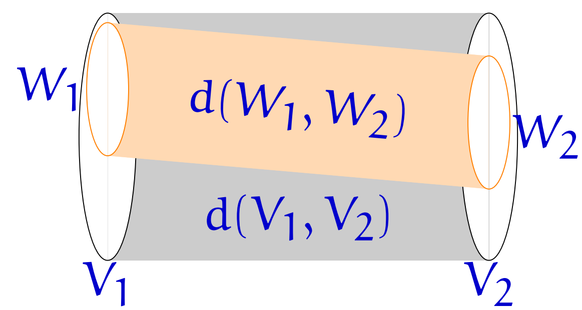
The regularity method

Theorem 4 (The Regularity Lemma (Szemerédi '78))**.** *For every* $\varepsilon > 0$ *and* $m \in \mathbb{N}$ *there is* $M \in \mathbb{N}$ *such that every graph* $G = (V, E)$ *can be partitioned into* $V = V_1 \cup \cdots \cup V_k$ *such that*

- ▷ $m \leq k \leq M$,
- ▷ $|V_1| \leq |V_2| \leq \cdots \leq |V_k| \leq |V_1| + 1$, *and*
- ▷ (V_i, V_j) *is* ε*-regular for at least* $(1 - \varepsilon)\binom{k}{2}$ *pairs* $ij \in \binom{[k]}{2}$.

The regularity method





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For every $\delta > 0$, Δ, $r \in \mathbb{N}$ *there is* $\varepsilon > 0$ *such that the following holds.*

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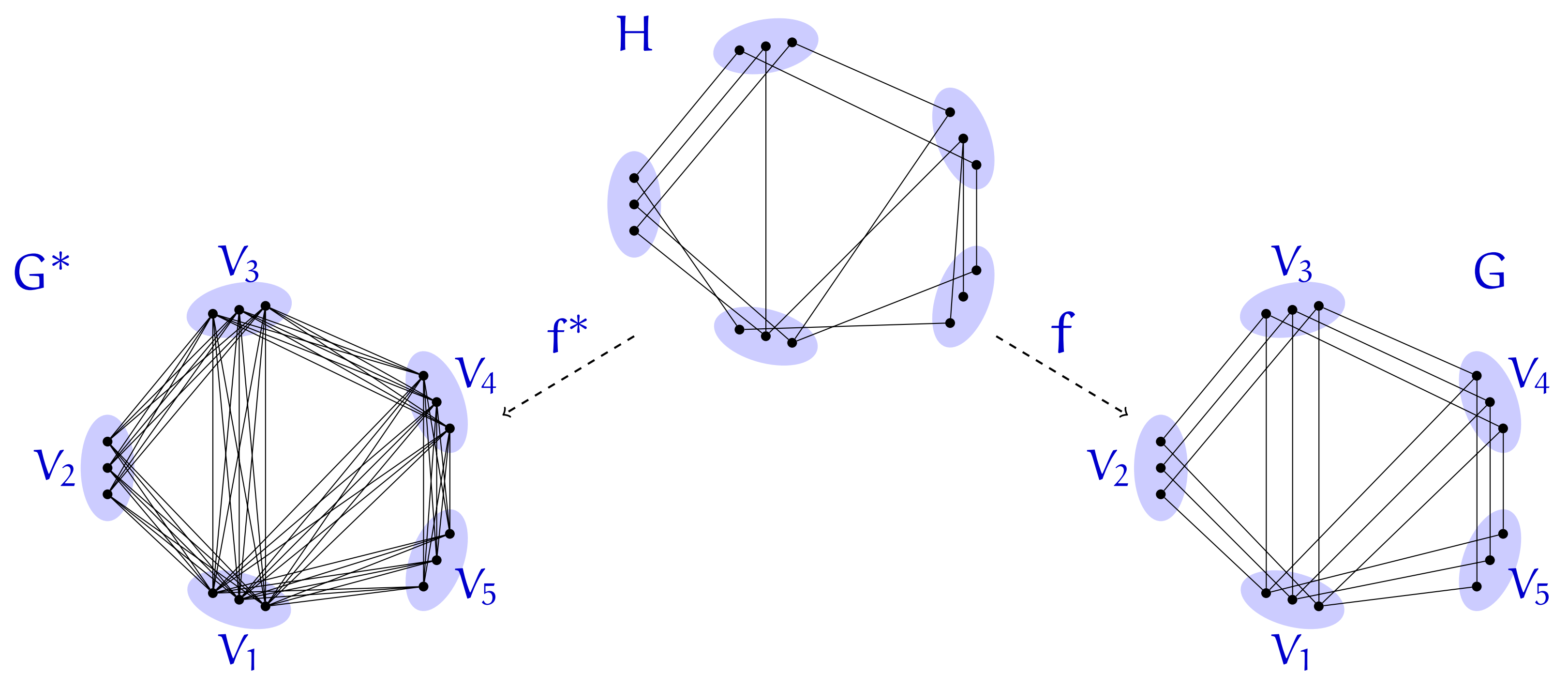

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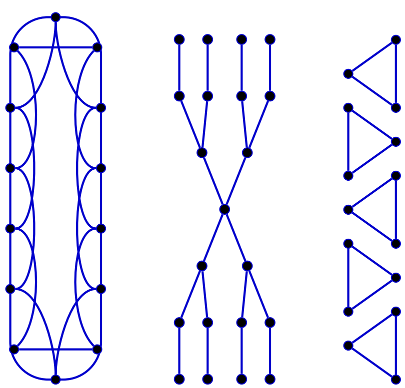

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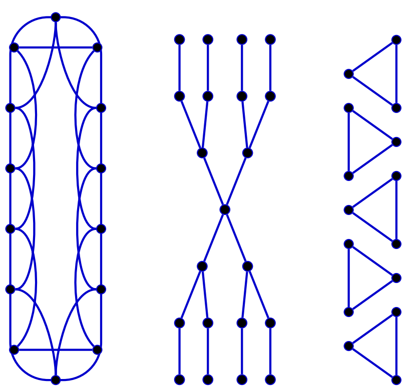

Answer: Yes!

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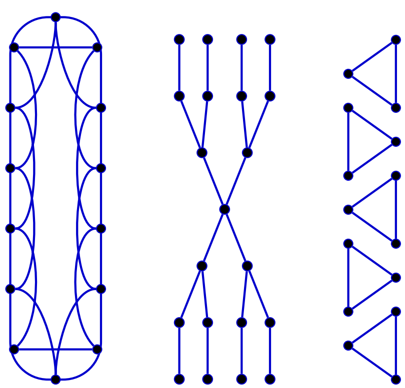

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Spanning subgraphs with constant maximum degree!

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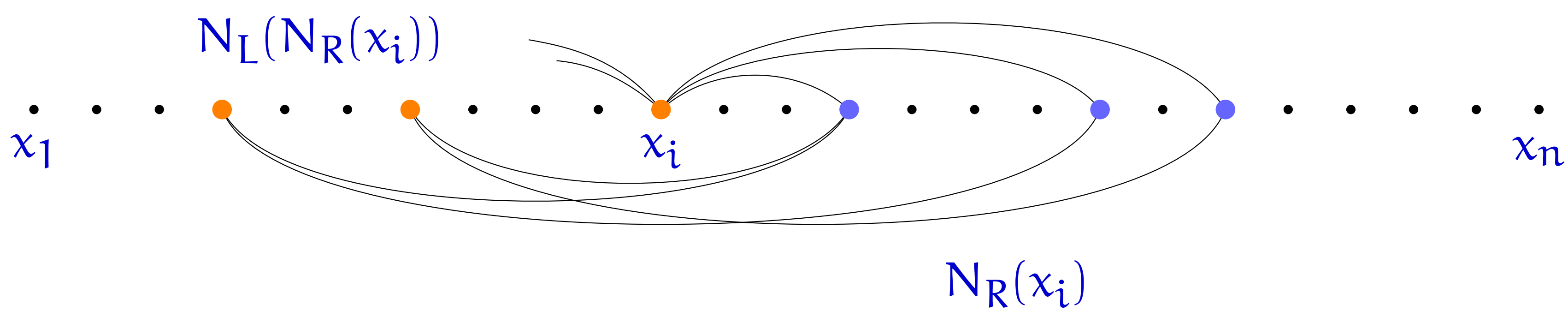
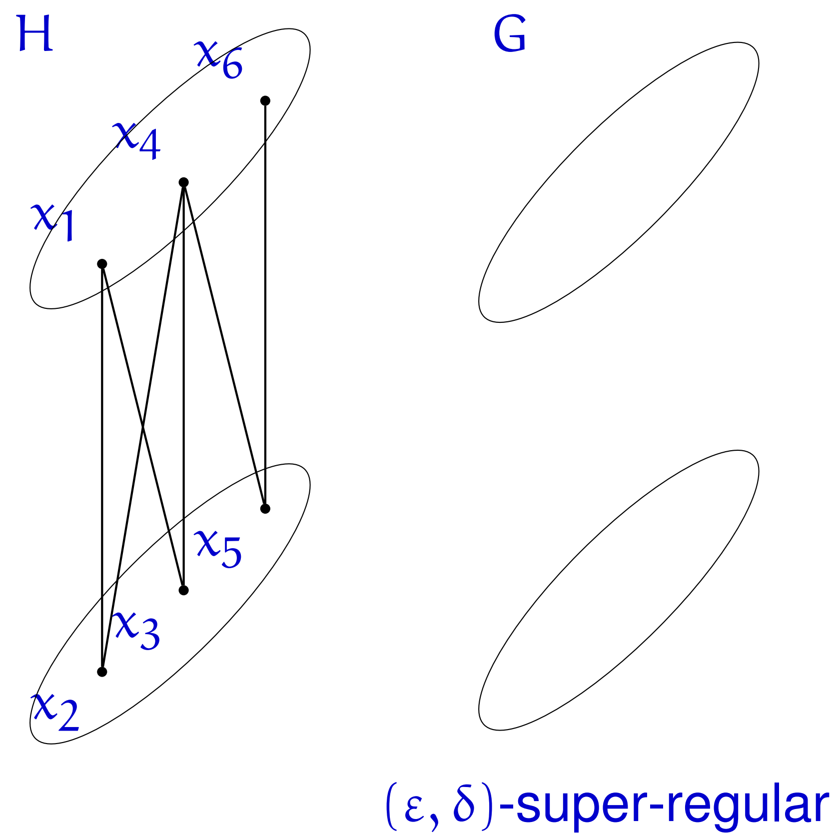


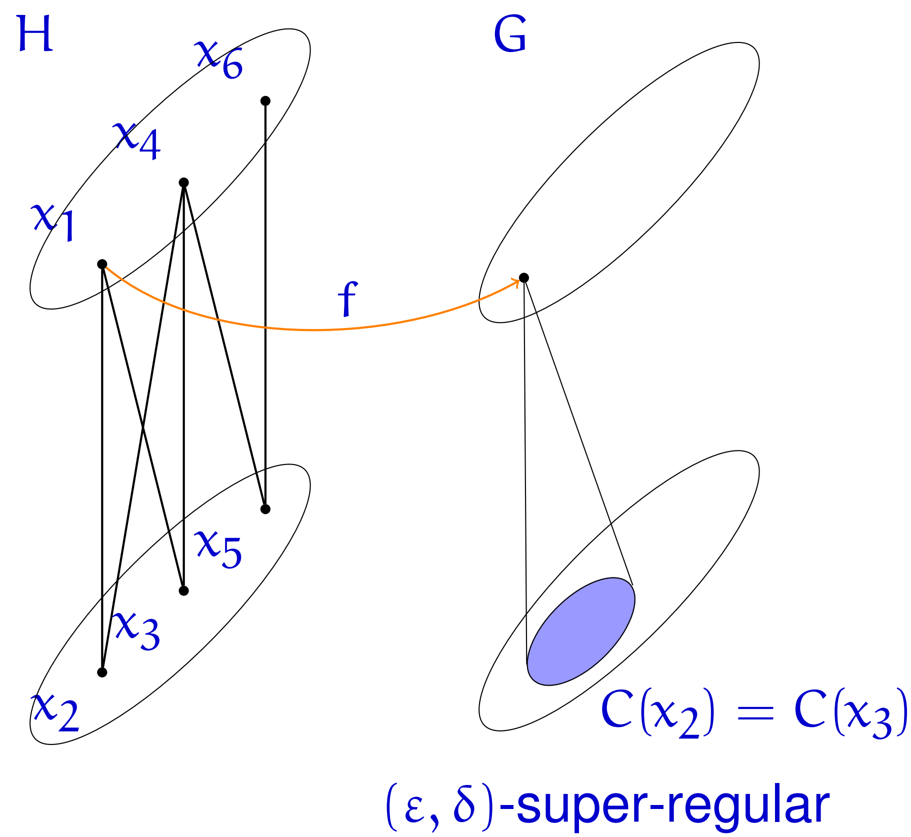
The embedding method

- ▷ randomized greedy embedding along the arrangeable ordering



The embedding method

- ▷ randomized; follow arrangeable ordering
- $C(x) = \bigcap_{y \in N_L(x)} N_G\big(f(y)\big)$
- guarantee candidate sets for successors



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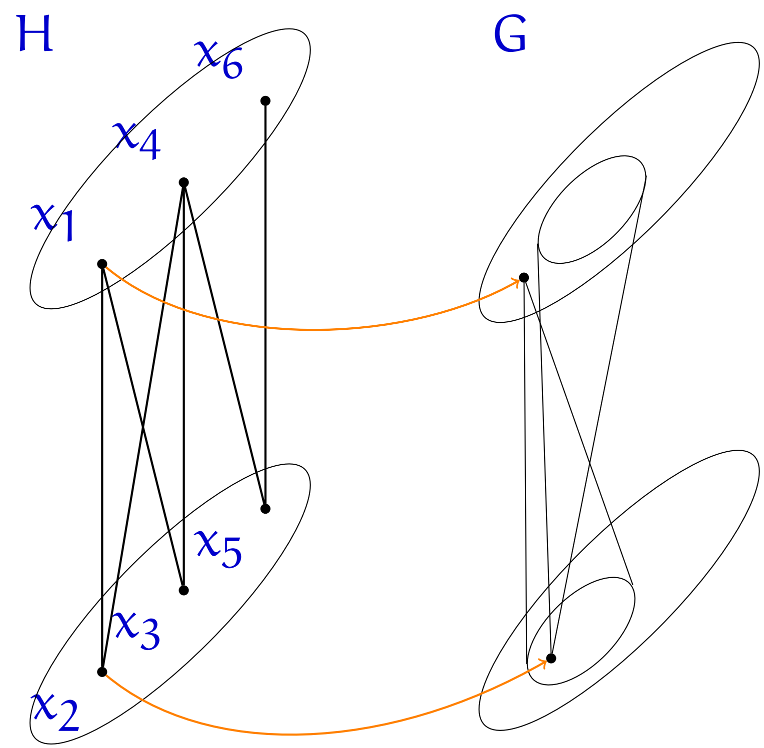

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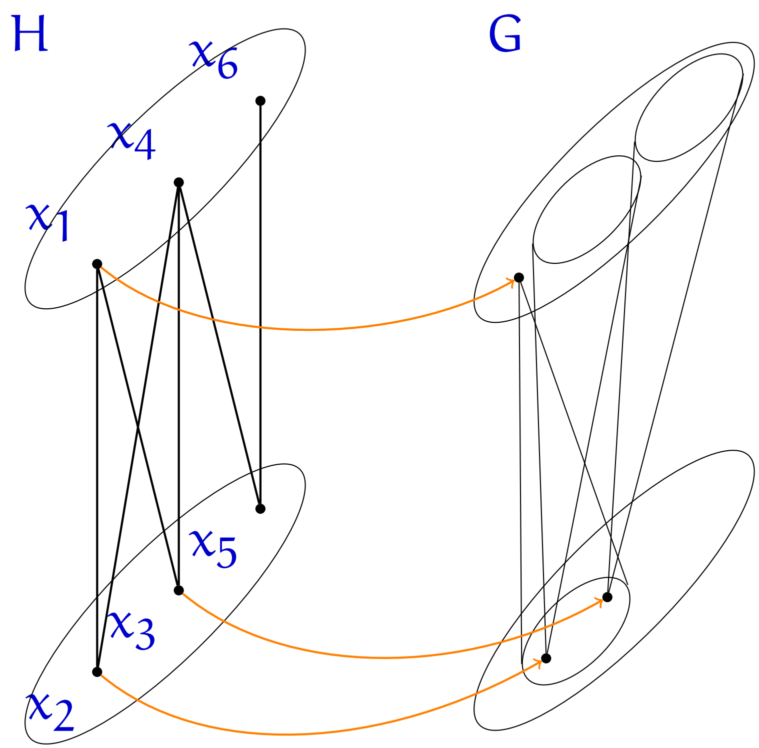

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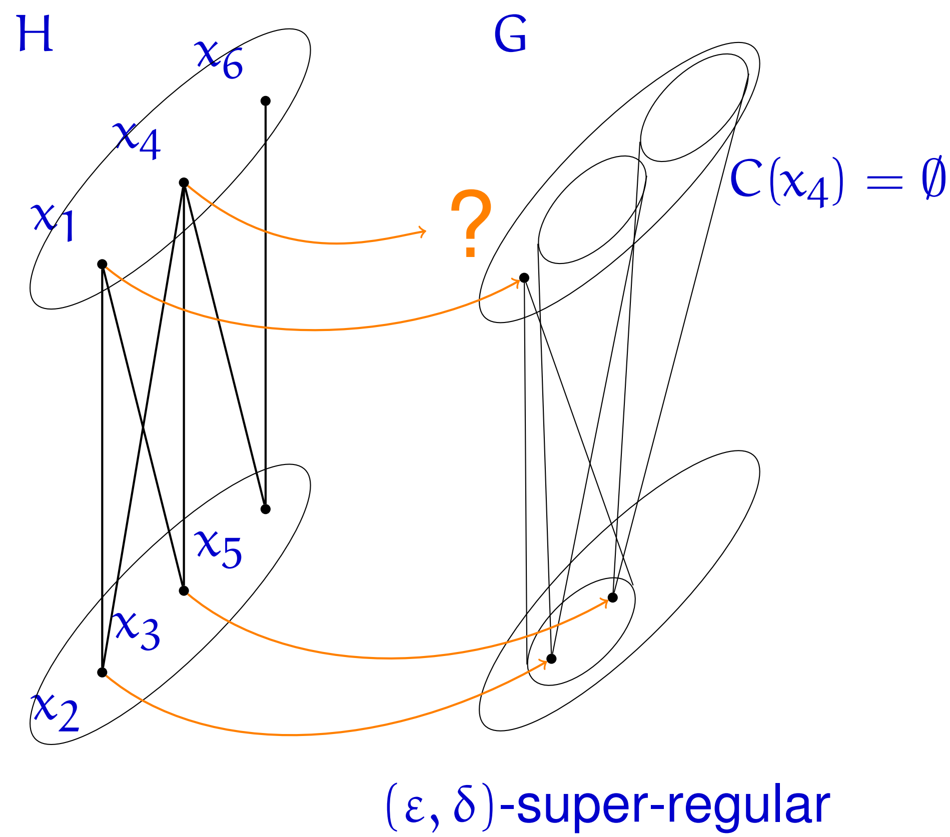

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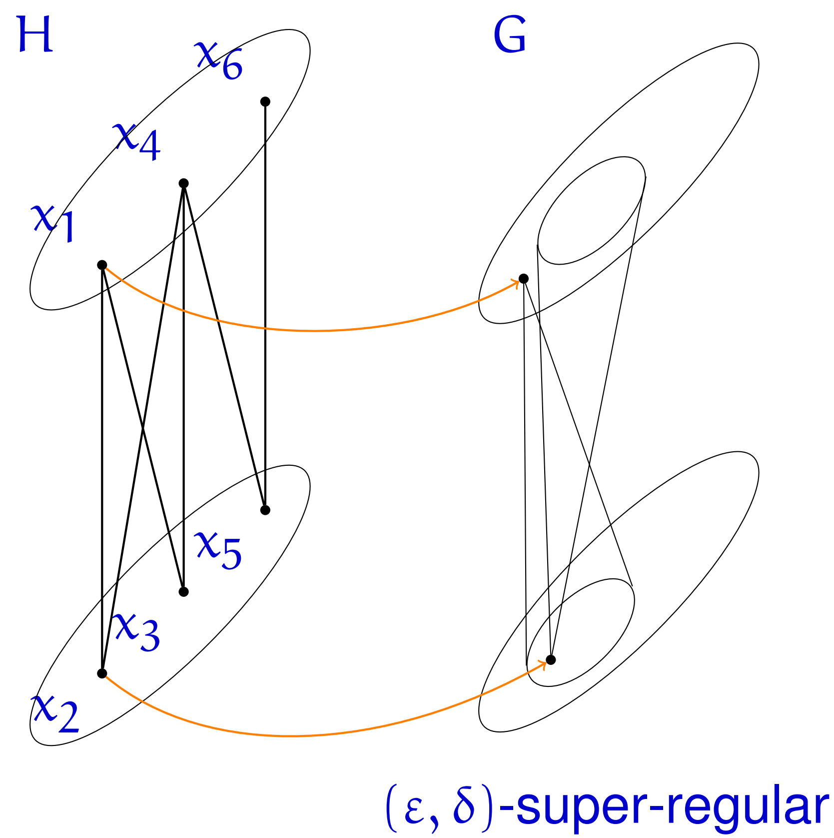
- respect ONE successor with the help of ε-regularity



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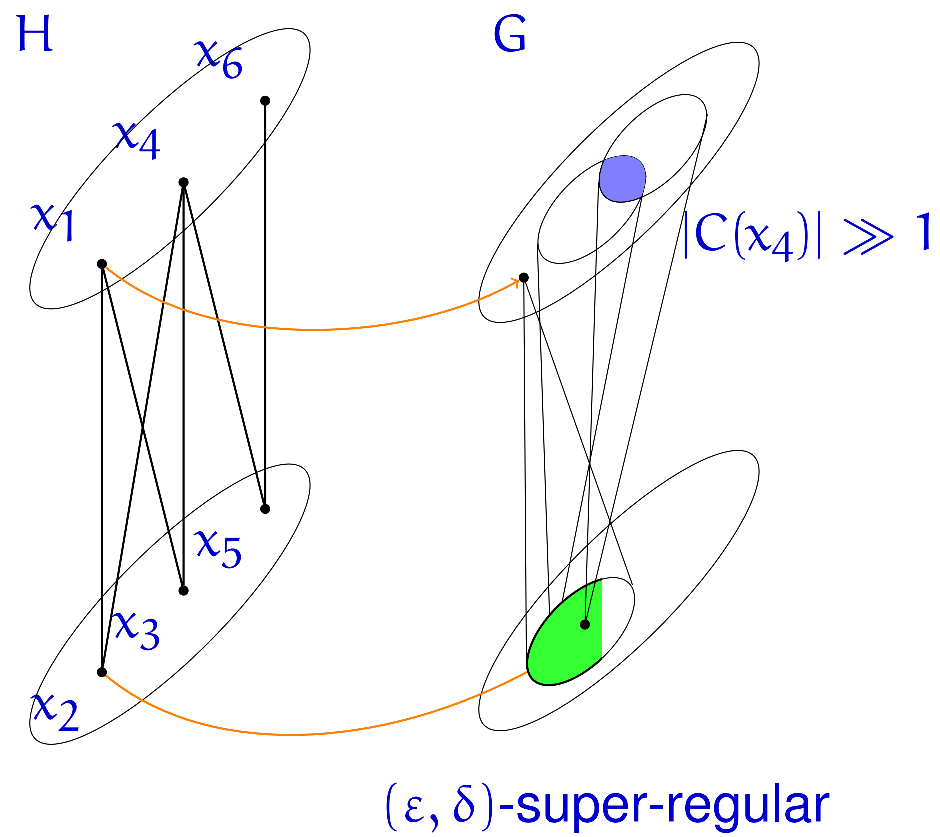
All but at most $2\varepsilon n$ vertices in $C(x_3)$ have the "correct" degree into $C(x_4)$.



Why does arrangeability help?

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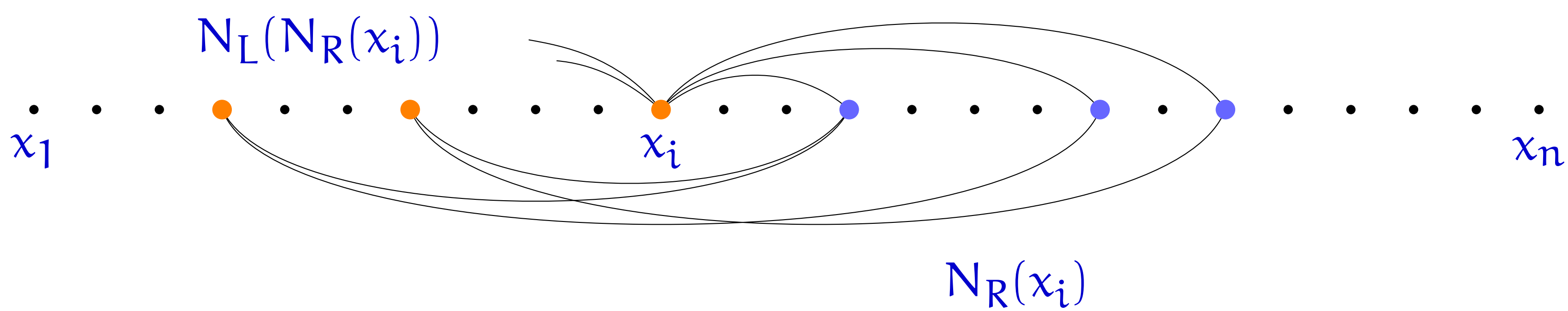
Problem: each successor might exclude $2\varepsilon n$ candidates

Solution: the a-arrangeability of H

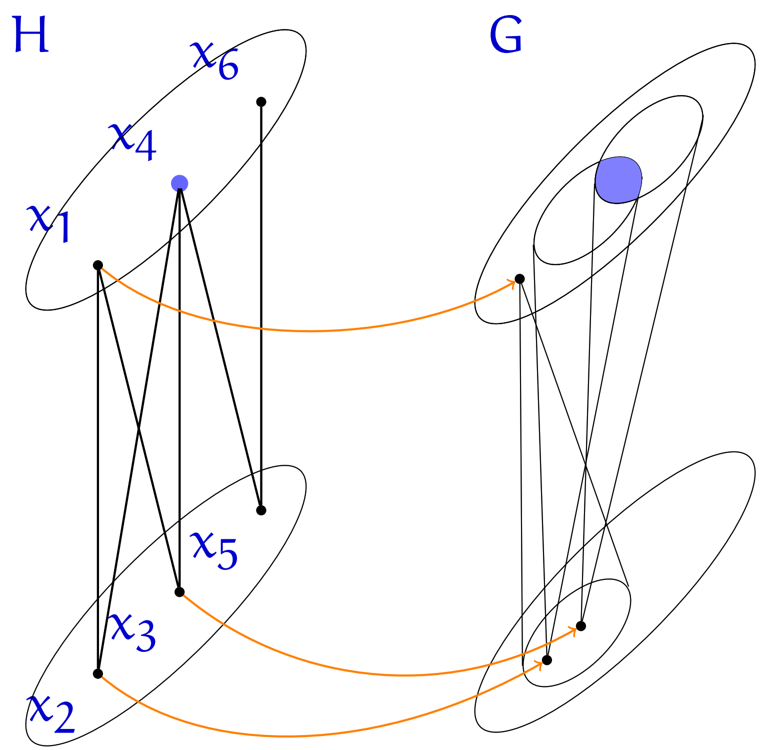


all successors of x_i have at most a predecessors in total

$\Rightarrow$ these share at most 2^a *different* candidate sets

$\Rightarrow$ we exclude at most $2^{a+1}\varepsilon n$ candidates

The embedding method



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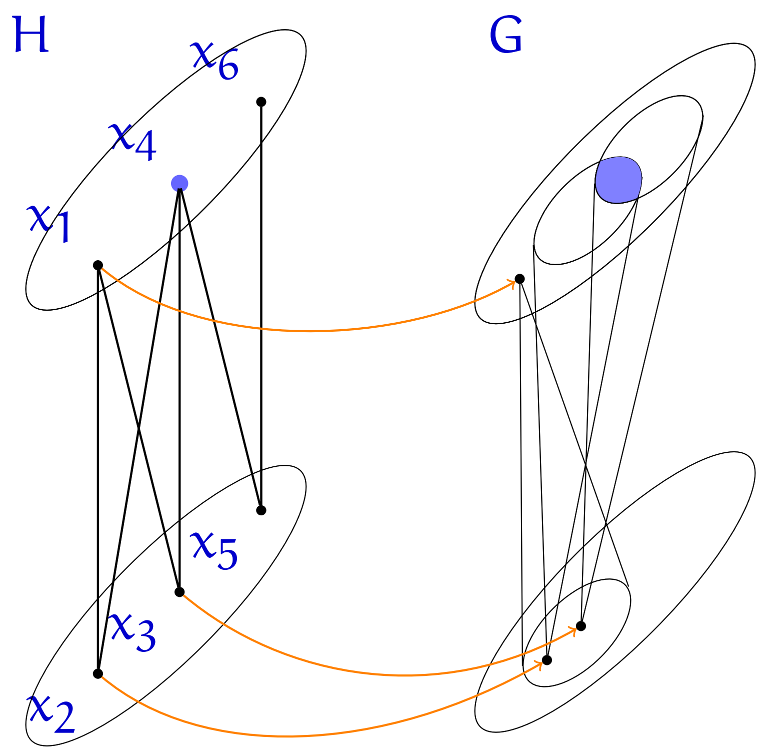

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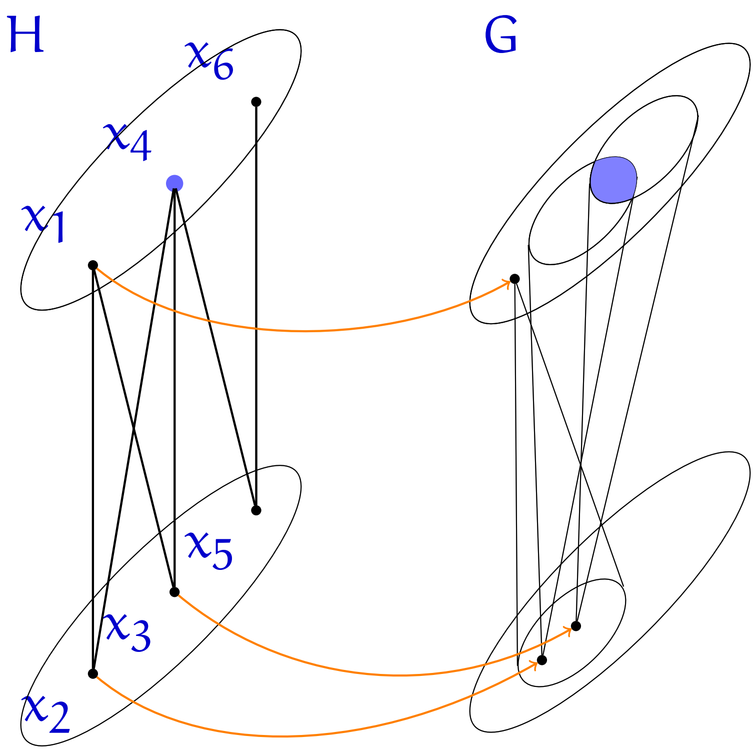

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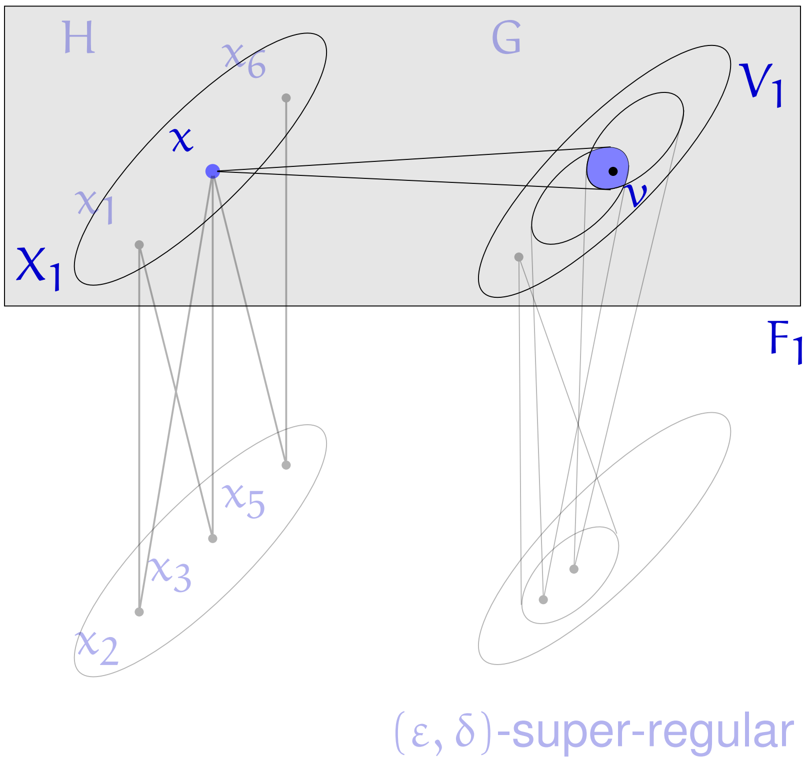
▷ handle occupied candidate sets

▷ finish the embedding with a König–Hall type argument

The auxiliary graphs



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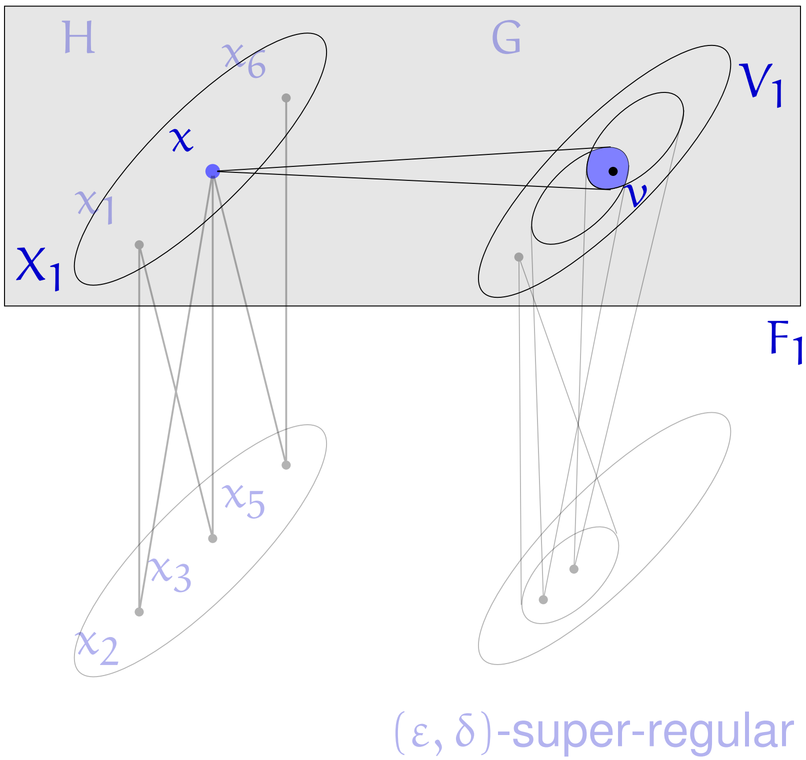

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The auxiliary graphs:
$F_i = (X_i \cup V_i, E_i)$ with $\{x, v\} \in E_i$ if and only if $v \in C(x)$.

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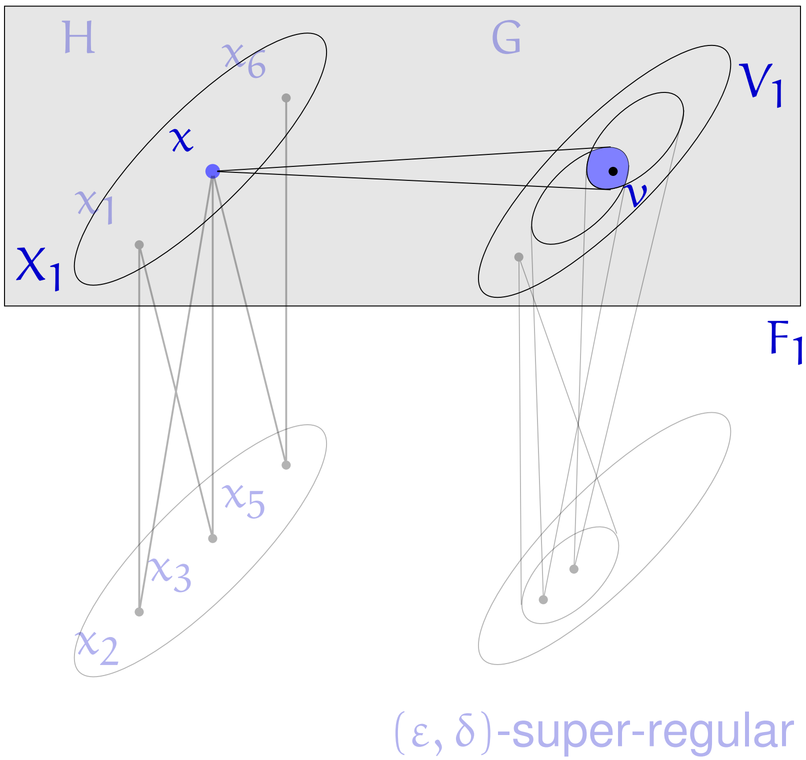

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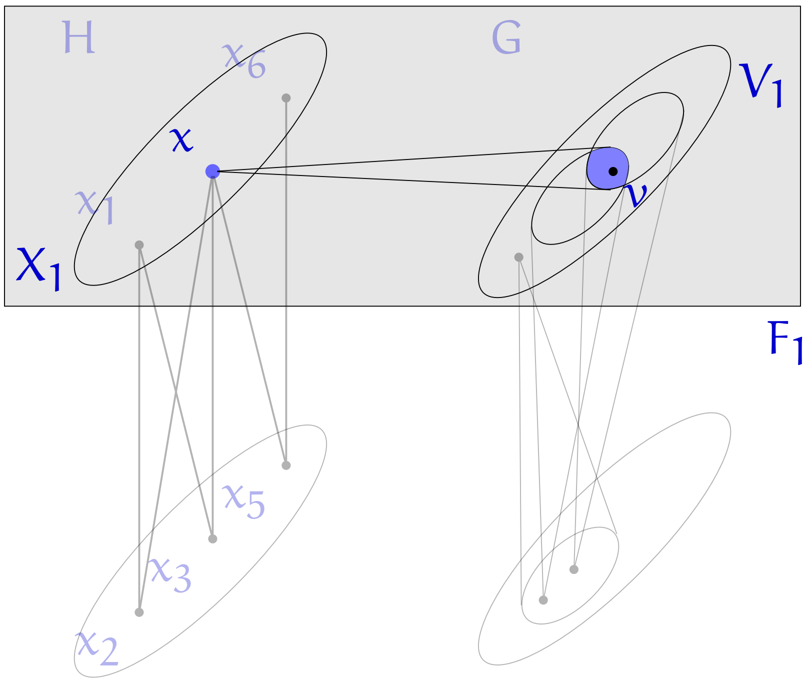

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An application in Ramsey theory

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Theorem 8 (Böttcher, Taraz & Würfl '13+)**.** *Almost every planar graph* H *is such that* $R(H) \leq 12\,|H|$*.*

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- ▷ **Böttcher, Taraz & Würfl** make use of the **arrangeable blow-up lemma** to obtain $R(H) \leq 12|H|$ for almost every planar H

Manuscripts

- ▷ http://arxiv.org/abs/1305.2059
- ▷ http://arxiv.org/abs/1305.2078