# Arithmetical functions with regular behaviour

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**Dedicated to P. Erdős on his 100th anniversary**

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## Notation

$\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathcal{P}$

$G$= Abelian group, $\mathcal{A}$, $\mathcal{A}^*$, $\mathcal{M}$, $\mathcal{M}^*$, $\mathcal{A}_G$, $\mathcal{A}_G^*$,

- $\mathcal{A}$ = set of additive functions
- $\mathcal{M}$ = set of multiplicative functions
- $\mathcal{A}^*$ = set of complete additive functions
- $\mathcal{M}^*$ = set of complete multiplicative functions.
- $f \in \mathcal{A}_G$ means: $f : \mathbb{N} \to G$, $f(nm) = f(n) + f(m)$ if $(n, m) = 1$
- $f \in \mathcal{A}_G^*$ means: $f : \mathbb{N} \to G$, $f(nm) = f(n) + f(m)$ holds for every $n, m \in \mathbb{N}$

# 1. On log $n$

Starting point: P. Erdős, On the distribution function of additive functions, *Ann.Math.*, **47** (1946), 1-20.
*$f \in \mathcal{A}$ and $\Delta f(n) := f(n+1) - f(n) \to 0 \ \ (n \to \infty)$ or $f(n) \leq f(n+1) \ \ (n \in \mathbb{N})$, then $f(n) = c \log n$.*

- $f \in \mathcal{A}$ and $\Delta^k f(n) \geq 0 \ \ (n \in \mathbb{N})$, then $f(n) = c \log n$
- $f \in \mathcal{A}$ and $\Delta f(n) \geq -K \ \ (n \in \mathbb{N})$, then $f(n) = c \log n + O(1)$, $u(n)$ is bounded (E. Wirsing)
- $f \in \mathcal{A}$ and $\frac{1}{x} \sum_{n \leq x} |\Delta f(n)| \to 0 \ \ (x \to \infty)$, then $f(n) = c \log n$
- $f \in \mathcal{A}$, $\exists \gamma$, $x_1 < x_2 < \cdots$ such that $\frac{1}{x_i} \sum_{x_i < n \leq \gamma x_i} |\Delta f(n)| \to 0 \ \ (i \to \infty)$, then $f(n) = c \log n$ (E. Wirsing)

- $f \in \mathcal{A}^*$ and $\frac{\Delta f(n)}{\log n} \to 0 \ \ (n \to \infty)$, then $f(n) = c \log n \ \ (E.Wirsing)$
- $f, g \in \mathcal{A}$ and $g(n+1) - f(n) \to 0 \ \ (n \in \infty)$, then $g(n) = f(n) = c \log n$
- $f, g \in \mathcal{A}$ and $\frac{1}{x} \sum_{n \le x} |g(n+1) - f(n)| \to 0 \ \ (x \in \infty)$, then $g(n) = f(n) = c \log n$
- If $P(x) \in \mathbb{R}[x]$, $E$ is the shift operator ($Ex_n = x_{n+1}$), $f \in \mathcal{A}$ and $\frac{1}{x} \sum_{n \le x} |P(E)f(n)| \to 0 \ \ (x \to \infty)$, then $f(n) = c \log n + u(n)$, where $P(E)u(n) = 0$ ($n \in \mathbb{N}$). If $P(1) \neq 0$, then $c = 0$, $u$ is of finite support, i.e. $u(p^\alpha) = 0$ for every large $p$ (P.D.T.A. Elliott - I. K.)
- If $f, g \in \mathcal{A}$ and $|g(n+1) - f(n) \le K$, then $f(n) = c \log n + h_1(n)$, $g(n) = c \log n + h_2(n)$, $h_1(n)$, $h_2(n)$ are bounded. (J. L. Mauclaire)

## I asked:

a) Characterize all those $f \in \mathcal{A}$ for which

$$f(an+b) - f(An+B) \to C \quad (n \to \infty),$$

and those $f, \ g \in \mathcal{A}$ for which

$$f(an+b) - g(An+B) \to C \quad (n \to \infty).$$

Partial results: I. K. , Bui Minh Phong (=BMP), J.-L. Mauclaire (JLM). Completely solved by Elliott (PDTAE).

b) Characterize all those $f_i \in \mathcal{A}$ $(i = 1, \cdots k)$ for which

$$\sum_{i=1}^{k} f_i(n+i) \to 0 \quad (n \to \infty). \tag{1.1}$$

**(C1) Conjecture 1.**

If (1.1) holds, then $f_i(n) = c_i \log n + u_i(n),\ u_i \in \mathcal{A}$ and

$$\sum_{i=1}^{k} u_i(n+i) = 0 \quad (n \in \mathbb{N}). \tag{1.2}$$

**(C2) Conjecture 2.**

If (1.2) holds, then

$$u_1(n) = \cdots = u_k(n) = 0 \ \text{ if } \ (n, (k-1)!) = 1.$$

Partial results: If (1.2) holds, $k \le 4,\ u_i \in \mathcal{A}^*$, then (C2) is true.
If (1.2) holds, $k \le 3,\ u_i \in \mathcal{A}$, then (C2) is true (R. Styer).

c) We say that a subset $\mathcal{B}$ of $\mathbb{N}$ is a set of uniqueness, if $f \in \mathcal{A}^*$, $f(b) = 0$ for all $b \in \mathcal{B}$ implies that $f(n) = 0$ for all $n \in \mathbb{N}$.

We say that a subset $\mathcal{B}$ of $\mathbb{N}$ is a set of uniqueness mod 1, if $f \in \mathcal{A}^*$, $f(b) = 0 \pmod 1$ $(\forall b \in \mathcal{B})$ implies that $f(n) = 0 \pmod 1$ for all $n \in \mathbb{N}$.

Let $\mathcal{P}_1 := \{p + 1 \mid p \in \mathcal{P}\}$.

**(C3) Conjecture 3.**

The set $\mathcal{P}_1$ is a set of uniqueness.

**(C4) Conjecture 4.**

The set $\mathcal{P}_1$ is a set of uniqueness $\pmod 1$.

**(C3):**

I. K. (almost), PDTA completely

**(C4):**

- I proved in (I.K., Acta Arith. 16 (1969/1970)): There is (an ineffective) $K$ such that every $n \in \mathbb{N}$ can be written as

$$n = a(n) \prod_{i=1}^{r} (p_i + 1)^{l_i},$$

where $l_i \in \mathbb{Z}$, and $P(a(n)) \le K$.

- Elliott proved in (Monatschrifte Math. 97 (1984), 85-97): The above assertion is true with $K = 10^{387}$.

d) What are those subsets $E \subseteq \mathbb{N}$ for which $f(e) \nearrow, \ (e \in E)$ implies that $f = c\log$?

Since $n(P_2 + 1) = (n+1)(P_2' + 1)$ has infinely many $P_2, P_2'$ solutions $(\omega(P_2) \le 2, \ \omega(P_2') \le 2)$, therefore $E = \{P_2 + 1\}$ is such a set.

**(C5) Conjecture 5**

If $f \in \mathcal{A}^*$ and $f(p+1) \ge f(q+1)$ for every couple of primes $p > q$, then $f(n) = c\log n \ (\forall n \in \mathbb{N})$.

**Elliott (Ramanujan J. (2008), 15, 87-102) proved:**

If $f \in \mathcal{A}$ and $f(p+1) \nearrow$ on $\mathcal{P}_1$, then $f(n) = c\log n$ for all odd $n$, and $f(2^\alpha) =$ constant.
Thus Conjecture 5 is true for $f \in \mathcal{A}^*$.

**(C6) Conjecture 6 (1977)**

If $f \in \mathcal{A}^*$ and $f(p+1) \geq 0$ for every $p \in \mathcal{P}$, then $f(n) \geq 0$ $(\forall n \in \mathbb{N})$.

This is hopless. It would follows from

**(C7) Conjecture 7**

For every $a \in \mathbb{N}$ there exists a $K$ for which $p+1 = Ka^n$ has infinitely many solutions as $n \in \mathbb{N}$.

**(C8) Conjecture 8 (A joint problem of BMP and mine)**

If $f_1, f_2 \in \mathcal{A}^*$ and

$$f_1(p+4) - f_2(p+2) \equiv 0 \pmod 1,$$

then $f_1(n) \equiv f_2(n) \equiv 0 \pmod 1$ $(n \in \mathbb{N})$.

# 2. Characterization of $n^s$ as a multiplicative function

**Question:** $f \in \mathcal{M}$, $\triangle f(n) = f(n+1) - f(n) \to 0 (n \to \infty)$. What further assumptions guarantee that $f(n) = n^s$, $s = \sigma + it$?

**Theorem 1. (I.K)**

Let $f, g \in \mathcal{M}$ and

$$\sum_{n=1}^{\infty} \frac{1}{n} |g(n+1) - f(n)| < \infty.$$

Then either $\sum_{n=1}^{\infty} \frac{1}{n} |f(n)| < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{n} |g(n)| < \infty$ or

$$f(n) = g(n) = n^{\sigma + it}, \ \sigma, t \in \mathbb{R}, \ 0 \le \sigma < 1.$$

## Theorem 2.(I.K)

Let $f, g \in \mathcal{M}^*$, $k \in \mathbb{N}$, $k \geq 2$ and

$$\sum_{n=1}^{\infty} \frac{1}{n} |g(n+k) - f(n)| < \infty.$$

Let $f(n) = g(n) = 0$ if $(n, k) > 1$ and $f(n) \neq 0$, $g(n) \neq 0$ if $(n, k) = 1$. Then either

$$\sum_{n=1}^{\infty} \frac{1}{n} |f(n)| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n} |g(n)| < \infty,$$

or there exist $F, G \in \mathcal{M}^*$, $s \in \mathbb{C}$, $\Re s < 1$ such that $f(n) = n^s F(n), \; g(n) = n^s G(n)$ and $G(n+k) = F(n)$ $(\forall n \in \mathbb{N})$ hold.

**Theorem 3.(I.K+BMP)**

Let $F, G \in \mathcal{M}$, $k \in \mathbb{N}$ and

$$G(n+k) = F(n) \ (\forall n \in \mathbb{N}).$$

Let

$$S_F := \{n \in \mathbb{N} \mid F(n) \neq 0\}, S_G := \{n \in \mathbb{N} \mid G(n) \neq 0\}.$$

Then either $S_F, S_G$ are finite sets, or $F(n) \neq 0, G(n) \neq 0$ for every $n \in \mathbb{N}, (n,k) = 1$.

## (C9) Conjecture 9.

If $f \in \mathcal{M}$ and

$$\frac{1}{x}\sum_{n\le x}|\Delta f(n)| \to 0 \text{ as } x\to\infty,$$

Then either

$$\frac{1}{x}\sum_{n\le x}|f(n)| \to 0 \text{ as } x\to\infty,$$

or

$$f(n) = n^s, \Re s < 1.$$

Hildebrand proved:

**Theorem 4. (Hildebrand)**

There exists a suitable $c$ with the following property: If $g \in \mathcal{M}^*$, $|g(n)| = 1,\ (n \in \mathbb{N})$, and $|g(p) - 1| \leq c$ $(\forall p \in \mathcal{P})$, then either $g(n) = 1$ identically, or

$$\liminf \frac{1}{x} \sum_{n \leq x} |\Delta g(n)| > 0 \text{ as } x \to \infty,$$

I proved

**Theorem 5. (I.K)**

Let $g \in \mathcal{M}^*$, $|g(n)| = 1,\ (n \in \mathbb{N})$. There exist positive constants $\beta < 1$ and $\delta$ such that

$$\limsup \sum_{x^\beta < p < x} \frac{|g(p) - 1|}{p} \leq \delta \text{ and } \liminf \sum_{\frac{x}{2} < p < x} |\Delta g(n)| = 0$$

imply that $g(n) = 1$ identically.

**Theorem 6., (Wirsing proved in 1984; another proof: Wirsing, Tang and Shao)**

If $g \in \mathcal{M}$, $|\Delta g(n)| \to 0$, then either $f(n) \to 0 (n \to \infty)$, or $f(n) = n^s,\ 0 \leq \Re s < 1$.

Another formulation of Wirsing's theorem:
Let $T$ be the additive group of real numbers $\pmod 1$.

**Theorem 7.**

If $F \in \mathcal{A}_T$, $\Delta F(n) \to 0 (n \to \infty)$, then $F$ is a restriction of a continuous homomorphism from $\mathbb{R}_x \to T$, that is $F(n) \equiv c \log n \pmod 1$.

**Theorem 8. (BMP)**

Let $A, B \in \mathbb{N}$, $D \in \mathbb{R}$ constant. If $h \in \mathcal{A}_T$ and

$$h(An + B) - h(n) - D \to 0 \quad \text{as} \quad n \to \infty,$$

then $h$ is a restriction of a continuous homomorphism from $\mathbb{R}_x \to T$.

**Theorem 9.( N.L.Bassily and I.K)**

If $f, \ g \in \mathcal{M}$ satisfy

$$g(2n+1) - Cf(n) = o(1) \ \ (n \to \infty)$$

with some non-zero constant $C$, then the following possibilities can occur:

(1) $f(n) \to 0, \ g(2n+1) \to 0 \ \ (n \to \infty)$

(2) $C = f(2), \ f(n) = n^s, \ g(m) = m^s, \ \ 0 \leq \text{Re } s < 1$ for all $n, \ m \in \mathbb{N}, \ (m, 2) = 1$.

(3) $C = f(2), \ f(n) = (-1)^{n-1} n^s, \ g(m) = \chi_4(m) m^s, \ \ 0 \leq \text{Re } s < 1$ for all $n, \ m \in \mathbb{N}, \ (m, 2) = 1$, where $\chi_4$ is the nonprincipal Dirichlet character $\pmod 4$.

We note that BMP proved that if $A, B \in \mathbb{N}$, $C \in \mathbb{C} \setminus \{0\}$, $f,\ g \in \mathcal{M}$ and

$$g(An+1) - Cf(n) = o(1) \quad \text{as} \quad n \to \infty$$

hold, then either $f(n) = o(1)$ and $g(An+1) = o(1)$ as $n \to \infty$ or there exist a complex number $s$ and functions $F,\ G \in \mathcal{M}$ such that $f(n) = n^s F(n),\ g(m) = n^s G(n),\ (0 \leq \Re s < 1)$ and $G(An+1) = \frac{1}{F(2)} F(n)$ are satisfied for all $n \in \mathbb{N}$.

**Problem**

Determine $f, g \in \mathcal{M}$ for which

$$g(An+B) - Cf(an+b) \to 0 \ \ (n \to \infty),$$

if $\frac{An+B}{an+b} \neq$ constant.

# 4. On additive function (mod 1)

Let $T = \mathbb{R}/\mathbb{Z}$ and $\mathcal{A}_T := \{F : \mathbb{N} \to T \mid F$ additive function$\}$.

**Definition.**

$F \in \mathcal{A}_T$ is of finite support if $F(p^\alpha) = 0$ for every large prime $p$, and every $\alpha \in \mathbb{N}$. Let $F_\nu \in \mathcal{A}_T$ $(\nu = 0, 1, \ldots, k-1)$ and

$$L_n(F_0, \ldots, F_{k-1}) = F_0(n) + \ldots + F_{k-1}(n+k-1) \; (n = 1, 2, \ldots).$$

Let $\mathcal{L}_0^{(k)}$ = space of those $(F_0, \ldots, F_{k-1}) \in \mathcal{A}_T^{(k)}$ for which

$$L_n(F_0, \ldots, F_{k-1}) = 0 \; (\forall n \in \mathbb{N}). \tag{3.1}$$

## (C10) Conjecture 10.

If (3.1) holds, then every $F_j$ is of finite support, and $\mathcal{L}_0^{(k)}$ is a finite dimensional $\mathbb{Z}$-module.

Let $G_j(n) = \tau_j \log n \pmod 1$, $\tau_0 + \ldots + \tau_{k-1} = 0$. Then

$$L_n(G_0, \ldots, G_{k-1}) \to 0 \text{ as } n \to \infty.$$

## (C11) Conjecture 11.

If $F_\nu \in \mathcal{A}_T \ (\nu = 0, 1, \ldots, k-1)$, and

$$L_n(F_0, \ldots, F_{k-1}) \to 0 \text{ as } n \to \infty,$$

then there are $\tau_0, \ldots, \tau_{k-1}$, $\tau_0 + \ldots + \tau_{k-1} = 0$ such that $F_j(n) = \tau_j \log(n) + H_j(n)$ $(j = 0, \ldots, k-1)$ and

$$L_n(H_0, \ldots, H_{k-1}) = 0 \ (\forall n \in \mathbb{N}).$$

### Remarks.

- 1) Conjecture 11 for $k = 1$ can be deduced from the theorem of Wirsing
- 2) Conjecture 11 for $k = 2$ under $F_\nu \in \mathcal{A}_T$ proved (I. K. 1984)
- 3) Conjecture 11 for $k = 2$ proved by R. Styer.
- 4) M. Wijsmuller investigated the analog question over the set of Gaussian integers.

**(C12) Conjecture 12.**

For every integer $k \geq 1$ there exists a constant $c_k$ such that for every prime $p > c_k$,

$$\min_{\substack{1 \leq j, \\ P(j) < p}} \max_{\substack{\ell \in [-k,k], \\ \ell \neq 0}} P(jp + \ell) < p$$

$$(\text{or} \quad \min_{1 \leq j < p} \max_{\ell \in [-k,k]} P(jp + \ell) < p).$$

It is open for $k \geq 2$.

## Theorem 10. (I.K)

Let $\mathcal{L}_0^{*(\ell)}$ = space of those $(F_0, \ldots, F_{\ell-1}) \in \mathcal{A}^{*}{}_T^{(\ell)}$ for which

$$L_n(F_0, \ldots, F_{k-1}) = 0 \ (\forall n \in \mathbb{N}).$$

Assume that Conjecture 12 is true for $k = \ell$. Then $\mathcal{L}_0^{*(\ell)}$ is a finite dimensional space.

Let $K$ = closure of $\{L_n(F_0, \ldots, F_{k-1}) | n \in \mathbb{N}\}$.

## (C13) Conjecture 13.

If $F_0, \ldots, F_{k-1} \in \mathcal{A}^{*}{}_T$ and $K$ contains an element of infinite order, then

$$K = T.$$

## (C14) Conjecture 14.

Let $f \in \mathcal{A}^*{}_T$, $\mathcal{H} = \{\alpha_1, \ldots, \alpha_k\}$ be the set of limit points of $f(n+1) - f(n)$ $(n \in \mathbb{N})$. Then

$$\mathcal{H} = E_k = \left\{\frac{m}{n} \pmod 1, m = 0, 1, \ldots k-1\right\},$$

furthermore

$$f(n) = \tau \log n + U(n) \pmod 1, \; U(\mathbb{N}) = E_k,$$

and for every $\omega \in E_k$ there exists a sequence $n_1 < n_2 < \ldots$ such that

$$U(n_\nu + 1) - U(n_\nu) \to 0 \;\; (n \to \infty).$$

**Theorem 11. (I. K. and M. V. Subbarao)**

- a) Conjecture 14 is true for $k = 1, 2, 3$.
- b) If $k = 4$, then either $4U(n) \equiv 0 \pmod 1$ or $5U(n) \equiv 0 \pmod 1$.

**Theorem 12. (E. Wirsing)**

Assume that the conditions of Conjecture 14 hold. Then

$$f(n) = \tau \log n + U(n) \pmod 1,$$

and there exists a finite $M \in \mathbb{N}$ such that $MU(n) \equiv 0 \pmod 1$.

**Theorem 13. (BMP)**

a) If a function $F \in \mathcal{M}^*$ and a positive integer $\ell \leq 5$ satisfy the condition $F(\mathbb{N}) = U_\ell := \{\omega \in \mathbb{C} | \omega^\ell = 1\}$, then $\{F(n+1)\overline{F}(n) \mid n \in \mathbb{N}\} = U_\ell$.

b) Assume that $A,\ B \in \mathbb{N}$, $G$ is any Abelian group and $F_1,\ F_2$ are $G$-valued completely multiplicative functions. If

$$\{F_1(An+B)(F_2(An))^{-1} \mid n \in \mathbb{N}\}$$

is a finite set, then there are finite subgroups $G_1$ and $G_2$ of $G$ such that $G_2 \subseteq G_1$ and $F_i(\mathbb{N})$ is a subgroup of $G_i$ (i=1,2).

# 5. Characterization of continuous homomorphisms as elements of $\mathcal{A}_G$ for compact groups.

**Definitions.**

$G$=metrically compact Abelian group,
$\rho =$ translation invariant metric
$k_D = \{$ those $\{x_n\} \in G^{\mathbb{N}}$ for which: if $x_{n_1}, x_{n_2}, \ldots$ is a convergent subsequence, then $x_{n_1+1}, x_{n_2+1}, \ldots$ is convergent also $\}$
$k_\Delta = \{\{x_n\} \in G^{\mathbb{N}} \mid \Delta x_n = x_{n+1} - x_n \to 0, \ (n \to \infty)\}$

Clear: $k_\Delta \subseteq k_D$.

$$\mathcal{A}_G^*(D) = \{f \in \mathcal{A}_G^* \mid \{f(n)\}_{n=1}^{\infty} \in k_D\}$$
$$\mathcal{A}_G^*(\Delta) = \{f \in \mathcal{A}_G^* \mid \{f(n)\}_{n=1}^{\infty} \in k_\Delta\}$$

## Results (Z. Daróczy and I.K.)

- 1) $\mathcal{A}_G^*(D) = \mathcal{A}_G^*(\Delta)$
- 2) If $f \in \mathcal{A}_G^*(D)$, then there exists a continuous homomorphism $\Phi : \mathbb{R}_x \to G$ such that $f(n) = \Phi(n)$ $(n \in \mathbb{N})$. It was deduced from the theorem of Wirsing (Theorem 6).
- 3) Let $S_f$= closure $\{f(1), f(2), \ldots\}$. Then $S_f$ is a compact subgroup of $G$.
- 4) $f \in \mathcal{A}_G^*(D)$ if and only if there exists a continuous function $H_f : S_f \to S_f$ such that

$$f(n+1) - H(f(n)) \to 0 \ (n \to \infty).$$

- 5) Bui Minh Phong proved further interesting results.

Thank you for your attention!