Resolvability of topological spaces

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DEFINITION. (Hewitt, 1943, Pearson, 1963)

– A topological space X is κ-resolvable iff it has κ disjoint dense subsets. (resolvable $\equiv$ 2-resolvable)

– X is maximally resolvable iff it is $\Delta(X)$-resolvable, where

$$\Delta(X) = \min\{|G| : G \neq \emptyset \text{ open in } X\}.$$

EXAMPLES:

– $\mathbb{R}$ is maximally resolvable.

– Compact Hausdorff, metric, and linearly ordered spaces are maximally resolvable.

QUESTION. What happens if these properties are relaxed?

Malychin's problem

EXAMPLE. (Hewitt, '43) There is a countable T_3 space X that is
– crowded (i.e. $\Delta(X) = |X| = \aleph_0$) and
– irresolvable($\equiv$ not 2-resolvable).

PROBLEM. (Malychin, 1995)

Is a Lindelöf T_3 space X with $\Delta(X) > \omega$ resolvable?

NOTE. Malychin constructed Lindelöf irresolvable Hausdorff (= T_2) spaces, and Pavlov Lindelöf irresolvable Uryson (= $T_{2.5}$) spaces.

THEOREM. (Filatova, 2004)

YES, every Lindelöf T_3 space X with $\Delta(X) > \omega$ is 2-resolvable.

This is the main result of her PhD thesis. It didn't work for 3 !

Pavlov's theorems

$$s(X) = \sup\{|D| : D \subset X \text{ is discrete}\}$$
$$e(X) = \sup\{|D| : D \subset X \text{ is closed discrete}\}$$

THEOREM. (Pavlov, 2002)

(i) Any T_2 space X with $\Delta(X) > s(X)^+$ is maximally resolvable.
(ii) Any T_3 space X with $\Delta(X) > e(X)^+$ is ω-resolvable.

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THEOREM. (J-S-Sz, 2007)

Any space X with $\Delta(X) > s(X)$ is maximally resolvable.

THEOREM. (J-S-Sz, 2012)

Any T_3 space X with $\Delta(X) > e(X)$ is ω-resolvable. In particular, every Lindelöf T_3 space X with $\Delta(X) > \omega$ is ω-resolvable.

J-S-Sz

THEOREM. (J-S-Sz, 2007)

If $\Delta(X) \geq \kappa = \text{cf}(\kappa) > \omega$ and X has no discrete subset of size κ then X is κ-resolvable.

THEOREM. (J-S-Sz, 2012)

If X is T_3, $\Delta(X) \geq \kappa = \text{cf}(\kappa) > \omega$ and X has no closed discrete subset of size κ then X is ω-resolvable.

NOTE. For $\Delta(X) > \omega$ regular these suffice. If $\Delta(X) = \lambda$ is singular, we need to do extra work.

For $\Delta(X) = \lambda > s(X)$ we automatically get that X is $< \lambda$-resolvable.

But now $\Delta(X) = \lambda > s(X)^+$, so we may use Pavlov's Thm (i).

For $\Delta(X) = \lambda > e(X)^+$ we may use Pavlov's Thm (ii).

$<\lambda$-resolvable

THEOREM. (J-S-Sz, 2006)

For any $\kappa \geq \lambda = \mathrm{cf}(\lambda) > \omega$ there is a dense $X \subset D(2)^{2^\kappa}$ with $\Delta(X) = \kappa$ that is $< \lambda$-resolvable but not λ-resolvable.

NOTE. This solved a problem of Ceder and Pearson from 1967. We used the general method of constructing $\mathcal{D}$-forced spaces.

THEOREM. (Illanes, Baskara Rao)

If $\mathrm{cf}(\lambda) = \omega$ then every $< \lambda$-resolvable space is λ-resolvable.

PROBLEM.

Is this true for each singular λ? How about $\lambda = \aleph_{\omega_1}$?

DEFINITION.

The space X is monotonically normal (MN) iff it is T_1 (i.e. all singletons are closed) and it has a monotone normality operator H that "halves" neighbourhoods :

H assigns to every $\langle x, U \rangle$, with $x \in U$ open, an open set $H(x, U)$ s. t.

(i) $x \in H(x, U) \subset U$,

and

(ii) if $H(x, U) \cap H(y, V) \neq \emptyset$ then $x \in V$ or $y \in U$.

FACT. Metric spaces and linearly ordered spaces are MN.

QUESTION. Are MN spaces maximally resolvable?

SD spaces

DEFINITION.

(i) $D \subset X$ is strongly discrete if there are pairwise disjoint open sets $\{U_x : x \in D\}$ with $x \in U_x$ for $x \in D$.

EXAMPLE: Countable discrete sets in T_3 spaces are SD.

(ii) X is an SD space if every non-isolated point $x \in X$ is an SD limit.

THEOREM. (Sharma and Sharma, 1988)

Every T_1 crowded SD space is ω-resolvable.

THEOREM. (DTTW, 2002)

MN spaces are SD, hence crowded MN spaces are ω-resolvable.

PROBLEM. (Ceder and Pearson, 1967)

Are ω-resolvable spaces maximally resolvable?

[J-S-Sz]

[J-S-Sz] $\equiv$ I. JUHÁSZ, L. SOUKUP AND Z. SZENTMIKLÓSSY, *Resolvability and monotone normality*, Israel J. Math., 166 (2008), no. 1, pp. 1–16.

DEFINITION. X is a DSD space if every dense subspace of X is SD. Clearly, MN spaces are DSD.

Main results of [J-S-Sz]

– If κ is measurable then there is a MN space X with $\Delta(X) = \kappa$ that is ω_1-irrresolvable.

– If X is DSD with $|X| < \aleph_\omega$ then X is maximally resolvable.

– From a supercompact cardinal, it is consistent to have a MN space X with $|X| = \Delta(X) = \aleph_\omega$ that is ω_2-irresolvable.

This left a number of questions open.

decomposability of ultrafilters

DEFINITION.

– An ultrafilter $\mathcal{F}$ is μ-descendingly complete iff for any descending μ-sequence $\{A_\alpha : \alpha < \mu\} \subset \mathcal{F}$ we have $\bigcap\{A_\alpha : \alpha < \mu\} \in \mathcal{F}$.

μ-descendingly incomplete is called μ-decomposable.

– $\Delta(\mathcal{F}) = \min\{|A| : A \in \mathcal{F}\}$.

– $\mathcal{F}$ is maximally decomposable iff it is μ-decomposable for all μ with $\omega \leq \mu \leq \Delta(\mathcal{F})$.

FACTS.

– Any "measure" is countably complete, hence ω-indecomposable.

– [Donder, 1988] If there is a not maximally decomposable ultrafilter then there is a measurable cardinal in some inner model.

– [Kunen - Prikry, 1971] Every ultrafilter $\mathcal{F}$ with $\Delta(\mathcal{F}) < \aleph_\omega$ is maximally decomposable.

[J-M] $\equiv$ I. Juhász and M. Magidor, *On the maximal resolvability of monotonically normal spaces*, Israel J. Math, 192 (2012), 637-666.

Main results of [J-M]

(1) TFAEV

– Every DSD space (of cardinality $< \kappa$) is maximally resolvable.

– Every MN space (of cardinality $< \kappa$) is maximally resolvable.

– Every ultrafilter $\mathcal{F}$ (with $\Delta(\mathcal{F}) < \kappa$) is maximally decomposable.

(2) TFAEC

– There is a measurable cardinal.

– There is a MN space that is not maximally resolvable.

– There is a MN space X with $|X| = \Delta(X) = \aleph_\omega$ that is ω_1-irresolvable.