The divisor function and divisor problem

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The definition of $d(n)$

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when $(m, n) = 1$, so that $d(n)$ is a *multiplicative arithmetic function*. Further $d(p^\alpha) = \alpha + 1$ for $\alpha \in \mathbb{N}$ and p a generic prime.

The general divisor function

In general

$$\zeta^k(s) = \sum_{n=1}^{\infty} d_k(n) n^{-s} \qquad (k \in \mathbb{N},\ \Re e\, s > 1),$$

where the (general) divisor function $d_k(n)$ represents the number of ways n can be written as a product of k factors, so that in particular $d_1(n) \equiv 1$ and $d(n) \equiv d_2(n)$. The Riemann zeta-function is $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ for $\Re e\, s > 1$, otherwise it is defined by analytic continuation.

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$$d_k(p^{\alpha}) = (-1)^{\alpha}\binom{-k}{\alpha} = \frac{k(k+1)\cdots(k+\alpha-1)}{\alpha!}$$

for primes p and $\alpha\in\mathbb{N}$.

Iterations of $d(n)$

Let, for $\mathrm{k} \in \mathbb{N}$ fixed,

$$\mathrm{d}^{(1)}(\mathrm{n}) := \mathrm{d}(\mathrm{n}),\ \mathrm{d}^{(\mathrm{k})}(\mathrm{n}) := \mathrm{d}\left(\mathrm{d}^{(\mathrm{k}-1)}(\mathrm{n})\right) \quad (\mathrm{k} > 1)$$

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This follows if one considers (p_j is the j-th prime)

$$n = 2^1 \cdot 3^2 \cdot 5^4 \cdot \ldots p_k^{p_k - 1}$$

and lets $k \to \infty$.

The work of Erdős and Kátai

Let ℓ_k denote the k-th *Fibonacci number*:

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$$d^{(k)}(n) > \exp\left((\log n)^{1/\ell_k - \varepsilon}\right)$$

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and if

$$N_j = \prod_{i=1}^{S_j} p_i^{r_i},$$

say, then

$$N_{j+1} = (p_1 \cdots p_{r_1})^{p_i - 1}(p_{r_i+1} \cdots p_{r_1+r_2})^{p_2-1} \\ \cdots (p_{r_1+\cdots r_{S_j-1}+1} \cdots p_{r_1+\cdots r_{S_j}})^{p_{S_j}-1}.$$

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Then one has $d^{(k)}(N_k) = 2^r$, and the proof reduces to finding the lower bound for r.

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This follows from

$$\log d(n) = \sum_{i=1}^{r} \log(\alpha_i + 1) \ll r \log\log r = \omega(n) \log\log \omega(n),$$

and the bound $(\omega(n) = \sum_{p|n} 1)$

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We use throughout the lecture the notation

$$f(x) \ll g(x) \Longleftrightarrow f(x) = O(g(x)) \Longleftrightarrow |f(x)| \leqslant Cg(x)\ (x \geqslant x_0).$$

New work

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which turned out to be only by a factor of $\log\log n$ (in the exponent) smaller than the true upper bound. Namely, in 2011 Y. Buttkewitz, C. Elsholtz, K. Ford and J.-C. Schlage-Puchta practically settled the problem of the maximal order of $d^{(2)}(n)$ by proving

$$\max_{n\leqslant x}\log d(d(n)) = \frac{\sqrt{\log x}}{\log\log x}\left(C + O\left(\frac{\log\log\log x}{\log\log x}\right)\right),$$

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where $C = 2.7958\ldots$ is an explicit constant.

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In 1992 A.I. conjectured

$$\sum_{n\leqslant x} d\bigl(n + d(n)\bigr) = Dx\log x + O(x) \quad (D > 0).$$

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$$\sum_{n\leqslant x} d\big(n + d(n)\big) = Dx\log x + O(x) \quad (D > 0).$$

I. Kátai [Math. Panonica 18(2007)] obtained this formula with the error term $O(x\log x/\log\log x)$.

Erdős's work on $d(n)$ in short intervals

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$d(n)$ sei der Anzahl der teiler von n. Folgende asymptotische Formel ist wohlbekannt:

$$(1) \qquad \sum_{n=1}^{x} d(n) = x\log x + (2C-1)x + O(x^\alpha), \quad \alpha = 15/46$$

(C ist die Eulersche Konstante).
(1) gilt wahrscheinlich für jedes $\alpha > 1/4$; diese alte Vermutung scheint aber sehr tief zu sein.

Remark

The function in the O-term is commonly denoted by $\Delta(x)$*, thus*

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Note that $15/46 = 0.32608\ldots$, due to H.-E. Richert (1952), can be replaced by M.N. Huxley's (2003) $131/416 = 0.31493\ldots$.

Theorem

Erdős: Es sei $h(x)$ eine beliebige wachsende Funktion, die mit x gegen ∞ strebt. Es sei

$$f(x) > (\log x)^{2\log 2 - 1} \exp\left(h(x)\sqrt{\log\log x}\right).$$

Dann gilt für fast alle x

$$(*) \qquad \sum_{n \leqslant f(x)} d(x+n) = (1+o(1))f(x)\log x \qquad (x \to \infty).$$

Diese Formel lässt sich nicht weiter verschärfen. Ist nämlich

$$f(x) = (\log x)^{2\log 2 - 1} \exp\left(c\sqrt{\log\log x}\right) \qquad (c > 0),$$

so gilt (*) nicht mehr für fast alle x.

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$$(8)\quad \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt \ll_{k,\varepsilon} T^{1+\varepsilon}\Big(1 + \sup_{T^{1+\varepsilon}<M\ll T^{k/2}} G_k(M;T)M^{-1}\Big),$$

if, for $M < M' \leqslant 2M$, $T^{1+\varepsilon} \leqslant M \ll T^{k/2}$,

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This bound (8) provides a direct link between upper bounds for the $2k$-th moment of $|\zeta(\frac{1}{2}+it)|$ and sums of $\Delta_k(x,h)$ over the shift parameter h, showing also the limitations of the method, where $\Delta_k(x,h)$ denotes the error term in the asymptotic formula for the sum $\sum_{n\leqslant x} d_k(n)d_k(n+h)$.

Namely one writes

$$\sum_{n\leqslant x} d_k(n)d_k(n+h) = x\,P_{2k-2}(\log x; h) + \Delta_k(x,h),$$

where it is assumed that $k \geqslant 2$ is a fixed integer, and $P_{2k-2}(\log x; h)$ is a suitable polynomial of degree $2k-2$ in $\log x$, whose coefficients depend on k and h, while $\Delta_k(x,h)$ is supposed to be the error term.

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This means that we should have

$$\Delta_k(x,h) \;=\; o(x) \quad as \quad x \to \infty,$$

but unfortunately this is not yet known to hold for any $k \geqslant 3$, even for fixed h, while for $k = 2$ there are many results.

New results

S. Baier, T.D. Browning, G. Marasingha and L. Zhao (2012) proved

$$(9) \qquad \sum_{h \leqslant H} \Delta_3(N; h) \ll_\varepsilon N^\varepsilon \left(H^2 + H^{1/2} N^{13/12}\right) \qquad (1 \leqslant H \leqslant N),$$

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and if $N^{1/3+\varepsilon}\leqslant H\leqslant N^{1-\varepsilon}$, then there exists $\delta=\delta(\varepsilon)>0$ for which

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Remark

Note that (9), in the interval $N^{1/6+\varepsilon} \leqslant H \leqslant N^{1-\varepsilon}$, gives gives an asymptotic formula for the averaged sum $\sum_{h\leqslant H} D_3(N, h)$.

Theorem (A.I. + Jie Wu, 2011)

For fixed $k \geqslant 3$ we have

$$(10) \qquad \sum_{h\leqslant H} \Delta_k(N;h) \ll_\varepsilon N^\varepsilon\left(H^2 + N^{1+\beta_k}\right) \qquad (1 \leqslant H \leqslant N),$$

where β_k is defined by

$$\beta_k := \inf\left\{ b_k \;:\; \int_1^X |\Delta_k(x)|^2\,dx \ll X^{1+2b_k} \right\}$$

and $\Delta_k(x)$ is the remainder term in the asymptotic formula for $\sum_{n\leqslant x} d_k(n)$.

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Remark

We have

$$\sum_{n \leqslant x} d_k(n) = x p_{k-1}(\log x) + \Delta_k(x),$$

Remark

where

$$p_{k-1}(\log x) = \operatorname*{Res}_{s=1} \left(\zeta(s)^k \frac{x^{s-1}}{s} \right),$$

so that $p_{k-1}(z)$ is a polynomial of degree $k-1$ in z, all of whose coefficients depend on k. In particular,

$$p_1(z) \ = \ z + 2\gamma - 1 \qquad (\gamma = -\Gamma'(1)).$$

Remark

It is known that $\beta_k = (k-1)/(2k)$ for $k = 2, 3, 4$, $\beta_5 \leqslant 9/20, \beta_6 \leqslant 1/2$, etc. and $\beta_k \geqslant (k-1)/(2k)$ for every $k \in \mathbb{N}$. It is conjectured that $\beta_k = (k-1)/(2k)$ for every $k \in \mathbb{N}$, and this is equivalent to the Lindelöf Hypothesis that $\zeta(\frac{1}{2} + it) \ll_\varepsilon (|t| + 1)^\varepsilon$.

From the Theorem we obtain, for $1 \leqslant H \leqslant N$,

$$\sum_{h \leqslant H} \Delta_3(N; h) \ll_\varepsilon N^\varepsilon \left(H^2 + N^{4/3}\right),$$

$$\sum_{h \leqslant H} \Delta_4(N; h) \ll_\varepsilon N^\varepsilon \left(H^2 + N^{11/8}\right),$$

$$\sum_{h \leqslant H} \Delta_5(N; h) \ll_\varepsilon N^\varepsilon \left(H^2 + N^{29/20}\right),$$

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Remark

Since it is known that $\beta_k < 1$ for any k, this means that the bound in (10) improves on the trivial bound $HN^{1+\varepsilon}$ in the range $N^{\beta_k+\varepsilon} \leqslant H \leqslant N^{1-\varepsilon}$. Our result thus supports the assertion that $\Delta_k(N; h)$ is really the error term in the asymptotic formula for $D_k(N, h)$, as given above. In the case when $k = 3$, we have an improvement on the result of Baier et al. when $H \geqslant N^{1/2}$.

The basic idea of proof is to start from

$$\sum_{h\leqslant H}\Delta_k(N,h) = \sum_{N<n\leqslant 2N} d_k(n)\sum_{h\leqslant H} d_k(n+h) - \sum_{h\leqslant H}\int_N^{2N}\mathfrak{S}_k(x,h)\,dx$$

$$= M_k(N,H) + R_k(N,H) - \sum_{h\leqslant H}\int_N^{2N}\mathfrak{S}_k(x,h)\,dx,$$

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$$M_k(N,H) := \sum_{N<n\leqslant 2N} d_k(n)\,\underset{s=1}{Res}\left(\zeta(s)^k\frac{(n+H)^s-n^s}{s}\right),$$
$$R_k(N,H) := \sum_{N<n\leqslant 2N} d_k(n)\big(\Delta_k(n+H)-\Delta_k(n)\big),$$

and use complex integration to estimate $M_k(N,h)$ and connect $R_k(N,H)$ to mean square estimates for $\Delta_k(x)$.

We have

$$M_k(N,H) = H\int_N^{2N}\Big(\operatorname*{Res}_{s=1}\zeta(s)^k x^{s-1}\Big)^2\,dx + O_\varepsilon\Big(H^2N^\varepsilon + NH^{\alpha_k+\varepsilon} + N^{1+\beta_k+\varepsilon}\Big)$$

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and

$$\sum_{h\leqslant H}\int_N^{2N} \mathfrak{S}_k(x,h)\,dx = H\int_N^{2N} \Big(\operatorname*{Res}_{s=1} \zeta(s)^k x^{s-1}\Big)^2\,dx + O_\varepsilon\Big(N^{1+\varepsilon}\Big).$$

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By completing the estimations one obtains the assertion of the theorem.

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Theorem

Suppose $\log^2 T \ll U \leqslant T^{1/2}/2$, $T^{1/2} \ll H \leqslant T$, *then we have*

$$\int_T^{T+H} \max_{0\leqslant u\leqslant U}\Big|\Delta(x+u)-\Delta(x)\Big|^2\,dx \ll HU\mathcal{L}^5 + T\mathcal{L}^4\log\mathcal{L}$$
$$+ H^{1/3}T^{2/3}U^{2/3}\mathcal{L}^{10/3}(\log\mathcal{L})^{2/3},$$

where $\mathcal{L} := \log T$.

The author and W. Zhai (2012) obtained several new results involving $\Delta(x+U)-\Delta(x)$, where $\Delta(x) \equiv \Delta_2(x)$ and $U = o(x)$.

Theorem

Suppose $\log^2 T \ll U \leqslant T^{1/2}/2$, $T^{1/2} \ll H \leqslant T$, *then we have*

$$\int_T^{T+H} \max_{0\leqslant u\leqslant U} \Big|\Delta(x+u)-\Delta(x)\Big|^2 dx \ll HU\mathcal{L}^5 + T\mathcal{L}^4 \log \mathcal{L} + H^{1/3}T^{2/3}U^{2/3}\mathcal{L}^{10/3}(\log \mathcal{L})^{2/3},$$

where $\mathcal{L} := \log T$.

This generalizes and sharpens a result of D.R. Heath-Brown & K.-M. Tsang (1994).

Theorem

Suppose T, U, H are large parameters and $C > 1$ is a large constant such that

$$T^{131/416+\varepsilon} \ll U \leqslant C^{-1}T^{1/2}\mathcal{L}^{-5}, \quad CT^{1/4}U\mathcal{L}^5 \log\mathcal{L} \leqslant H \leqslant T.$$

Then in the interval $[T, T+H]$ there are $\gg HU^{-1}$ subintervals of length $\gg U$ such that on each subinterval one has $\pm\Delta(x) \geqslant c_\pm T^{1/4}$ for some $c_\pm > 0$.

Thank you for your attention!