The Monster group is the largest sporadic finite simple group. It has order about 10^{54}, degree of min. perm. repr. about 10^{20} and its smallest matrix repr. has

dim 196,883

The Monster group is remarkable and mysterious in many ways. Evidence for its existence was given by B. Fischer and independently by R. Griess in 1973 and it was constructed by R. Griess in 1980.

B. Fischer predicted it as a group generated by a conjugacy class (called 2A) of involutions, such that for $\tau, \pi \in 2A$ the M-orbit

$$\{(\tau^m, \pi^m) \mid m \in M\} \text{ is}$$

uniquely determined by the $\{(\tau\cdot\pi)^m \mid m \in M\}$, i.e. by the conjugacy class of the product and there are nine classes where the product $\tau\cdot\pi$ can be:

Alexander A. Ivanov

'Majorana representation of the Monster'

(a_τ, a_π)	Class of $\tau\cdot\pi$	$\dim \langle\langle a_\tau, a_\pi \rangle\rangle$		Centralizer in M of (τ, π)
1	1A	1		$2\cdot B$ [Fischer's Baby Monster]
$\frac{1}{8}$	2A	3		$2^2\cdot{}^2E_6(2)$
0	2B	2		$2^2\cdot 2^{22}\cdot Co_2$ [Conway group]
$\frac{13}{256}$	3A	4	u_ρ	Fi_{23} [Fischer 3-transposition]
$\frac{1}{64}$	3C	3		Th [Thompson group]
$\frac{1}{32}$	4A	5	v_ρ	$2^{1+22}\, McL$ [MacLaughlin group]
$\frac{1}{64}$	4B	5		$2\cdot F_4(2)$
$\frac{1}{128}$	5A	6	w_ρ	HN [Harada-Norton group]
$\frac{5}{256}$	6A	8	u_ρ	$2\cdot Fi_{22}$ [Fischer 3-transposition]

There is an invariant algebra $\cdot$ and inner products $(\ ,\ )$ on a 196,883-dimensional vector space V° such that $M = \mathrm{Aut}(V^\circ, \cdot, (\ ,\ ))$

This algebra played a crucial role in Griess construction of the Monster. By many reasons it is important to extend V° by 1-dimensional direct summand to obtain $V = V^\circ \oplus 1$ of dim 196,884 which is the linear term of modular invariant $J(q)$ McKay's observa.

2A-involutions and 196,884-algebra

Acting via conjugation on the 2A-involutions M realizes its minimal permutation representation

$$m: \tau \longmapsto m^{-1}\tau m, \quad m \in M, \tau \in 2A$$

of degree about 10^{20} and rank 9. The minimal module V^{0} is an irreducible constituents of the permutation module [the latter is multiplicity-free]. Thus there is a unique (up to rescaling) orthogonal proj:

$$\varphi^{0}: 2A \longrightarrow V^{0} = V_{196,883}$$

The algebra and inner products are projections of

$$(\tau, \pi) = \delta_{\tau\pi} \quad ; \tau \cdot \pi = \delta_{\tau\pi}\tau$$

$V = V_{196,884}$ is extension of V^{0} by 1-dim. trivial module, which is direct summand as a vector space but not an orthogonal complement or radical, although it is the identity of the extended algebra:

$$\varphi: 2A \longrightarrow V = V_{196,884}$$

Define $a_{\tau} := \varphi(\tau)$ [2A- or Majorana-axes]

(a_{τ}, a_{ρ}) and $\ll a_{\tau}, a_{\rho} \gg$ are determined by the class $(\tau\cdot\rho)^{M}$ and information is in the table

$V_{196,884}$ and Vertex operator algebras.

$$V^\natural = \underset{\dim 1}{V_0} + \underset{0}{V_1} + \underset{V}{V_2} + V_3 + \dots$$

$$j(q) = \frac{1}{q} + 744 + \sum_{i=2}^{\infty} \dim(V_i)\cdot q^{i-1} \quad \text{modular invar.}$$

Frenkel-Lepowsky-Meurman. $\mathrm{Aut}(V^\natural, \overset{\text{infinite number of products}}{\dots}) \cong M$

In this context a_τ is a conformal vector of central charge $\frac{1}{2}$ (also called Ising vector in VOA context). Miyamoto involution: a_τ generates a Virasoro algebra $L(\frac{1}{2},0)$ which has three irreducible modules: $L(\frac{1}{2},0)$, $L(\frac{1}{2},\frac{1}{2})$, $L(\frac{1}{2},\frac{1}{16})$. Then

$$\tau_{(a_\tau)} = \begin{cases} 1 & \text{on } \langle a_\tau\rangle\text{-submodules } L(\frac{1}{2},0) \text{ or } L(\frac{1}{2},\frac{1}{2}); \\ -1 & \text{on } \langle a_\tau\rangle\text{-submodules isomorphic } L(\frac{1}{2},\frac{1}{16}) \end{cases}$$

can be defined in a more general setting but in the Monster case gives back the 2A-involution τ.

This works because of the fusion rules for the Virasoro algebra (of central charge $\frac{1}{2}$) representations proved in the physical context by Belyavin-Polyakov-Zamolotchikov and mathematically Dong-Mason-Zhu.

Sakuma's Theorem.

Based on earlier work of M. Miyamoto, S. Sakuma proved that there are nine possible algebras generated by two Ising vectors (conformal vectors of central charge $\frac{1}{2}$). The possible inner products and dimensions (at level 2) match the nine possibilities in the Monster algebra and they correspond~~ing~~ to the classes 1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A that is by the M-classes of the products of the corresponding Miyamoto involutions.

Back to finite dimensions.

The algebra $(V = V_{196,884}; \cdot, (\,,\,))$ were studied by J. Conway and S. Norton. They determined all the nine two- 2A-axes generated subalgebras $\langle\langle a_\tau, a_\pi \rangle\rangle$, $\tau, \pi \in 2A$; calculated the spectrum of the adjoint action

$$a_\tau : v \longmapsto a_\tau \cdot v$$

(the algebra is commutative but not associative). Subject to suitable rescaling the spectrum is

$$\mathcal{S} = \{ 1, 0, \tfrac{1}{4}, \tfrac{1}{32} \}$$

a_τ is an idempotent (of length 1) and it is the unique ~~elem~~ 1-eigenvector of its adjoint action, up to scalars.

τ acting on V negates every $\frac{1}{32}$-eigenvector and centralizes every other eigenvector; negating every $\frac{1}{4}$-eig. and centralizing 1- and 0- eigenvectors gives an aut. of $C_V(\tau) = V^{(0)} \oplus V^{(1)} \oplus V^{(\frac{1}{4})}$ with restricted products.

This gives (and basically equivalent) to the fusion rules for the eigenvectors of a_τ:

	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1	0	$\frac{1}{4}$	$\frac{1}{32}$
0	0	0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$0, \frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$1, 0, \frac{1}{4}$

$v \in V_a^{(\lambda)}$
$u \in V_a^{(\mu)}$

$v \cdot u = \sum_{\nu \in S(\lambda,\mu)} d \cdot w$

$w \in V_a^{(\lambda)}$

$S(\lambda,\mu)$ – the (λ,μ)-entry of the fusion rules matrix.

This is essentially the same as the fusion rule for the Ising vector = Virasoro algebra of central charge $\frac{1}{2}$.

What Conway - Norton could not calculate:

Let $A \cong A_5$ be a $(2A, 3A, 5A)$ – subgroup in the Monster; take the fifteen involutions in A (three will suffice but this way it is symmetrical)

Q. What is the algebra generated by the correspond. (fifteen) $2A$ – axes?

$$V_A := \langle\langle a_\tau \mid \tau \in A \rangle\rangle \qquad ?$$

S. Norton has calculated

$$\dim V_A = 26 = \underset{2A}{15} + \underset{3A}{10} + \underset{5A}{1}$$

Majorana axiomatics

Ising model equivalent to Majorana Fermion

informed by: Alexander O. Gogolin 1965 – 2011

We axiomatize $(V, \overset{G}{T}, \cdot, (\,,\,), \overset{\varphi,\ \psi}{)}$, V a real vector space, G is a finite group, T a union of conjugacy classes in G, $\cdot$ algebra, $(\,,\,)$ inner product

$$\varphi : t \mapsto a_t \,; \quad t \in T, \quad a_t \in V$$

$$\psi : G \to GL(V) \,; \quad a_{g^{-1}tg} = (a_t)^{\psi(g)}$$

For $a_t \in Im(\varphi)$

$$V = V_{a_t}^{(1)} \oplus V_{a_t}^{(0)} \oplus V_{a_t}^{(\frac{1}{4})} \oplus V_{a_t}^{(\frac{1}{32})}, \quad a_t \cdot v = \mu \cdot v \text{ if } v \in V^{(\mu)}$$

$\tau(a_t)$ $\quad +1 \quad +1 \quad +1 \quad -1$

an automorphism of $(V, \cdot, (\,,\,))$

$\delta(a_t)$ $\quad +1 \quad +1 \quad -1 \quad$ not defined

an automorphism of $(\underbrace{V^{(1)} \oplus V^{(0)} \oplus V^{\frac{1}{4}}}_{V^+}, \cdot/_{V^+}, (\,,\,)/_{V^+})$

These two automorphisms are equivalent to the fusion rules

This is

<u>a Majorana representation of G.</u>

Every 2A-generated subgroup of the Monster possesses a Majorana repr.

(Majorana version)

<u>Sakuma's Theorem</u>. There precisely nine Majorana representations of the dihedral groups [two order 2 elements always generate a dihedral group of order twice the order of their product] the corresponding algebras are the two-generated subalgebras of the Monster algebra (the 2A-axes)

Hence we have nine <u>Norton-Sakuma algebras</u>

A_5 - theorem AAI, Ákos Seress (1958–2013)

There are four Majorana representations of A_5
(Math. Z, 2012)

Shape	dimension	Subgroup in M
$(2A, 3A, 5A)$	26	A_5
$(2A, 3C, 5A)$	20	A_5
$(2B, 3C, 5A)$	21	$A_5 \times 2$
$(2B, 3A, 5A)$	46	$A_5 \times 2$

Ákos + Sergey Shpectorov (classified by S. Norton)

This matches the 2A-generated subgroups in the Monster isomorphic to A_5 or to $A_5 \times 2$;
the possibility of the centre which does not act on $\langle\langle \mathrm{Im}(\varphi) \rangle\rangle$ was overlooked and pointed out by Ákos.