Combinatorial Dichotomy Classifications

Pavol Hell, SFU

Erdős Centennial, July 3, 2013

A homomorphism $G \to H$

An H-colouring of G

$f : V(G) \to V(H)$ with $uv \in E(G) \implies f(u)f(v) \in E(H)$

G, H are digraphs (undirected graphs are symmetric digraphs)

K_n-colouring = n-colouring

A homomorphism $G \to H$

An H-colouring of G

$f : V(G) \to V(H)$ with $uv \in E(G) \implies f(u)f(v) \in E(H)$

G, H are digraphs (undirected graphs are symmetric digraphs)

K_n-colouring = n-colouring

Example $H = C_5$

C_5-colourability is NP-complete

A homomorphism $G \to H$

An H-colouring of G

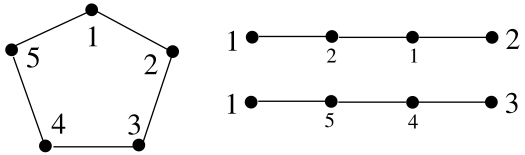
$f : V(G) \to V(H)$ with $uv \in E(G) \implies f(u)f(v) \in E(H)$

G, H are digraphs (undirected graphs are symmetric digraphs)

K_n-colouring = n-colouring

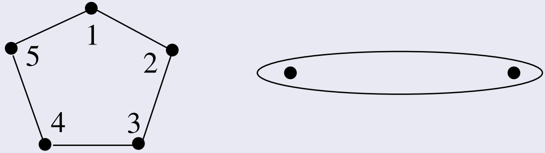
Example $H = C_5$

C_5-colourability is NP-complete



Gadgets

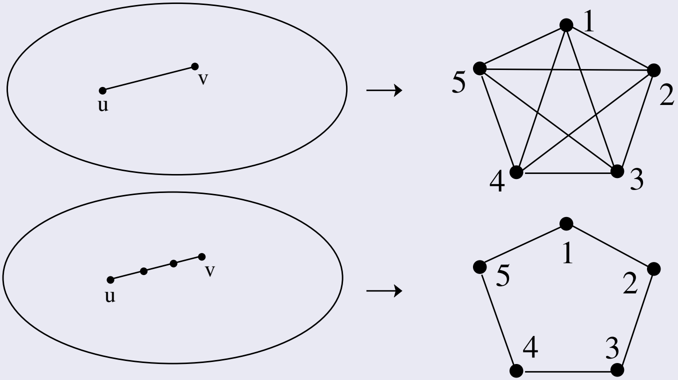
Possible colours $1, 2, 3, 4, 5$



- Any C_5-colouring assigns different possible colours
- Any assignment of different possible colours extends to the whole gadget

Gadgets

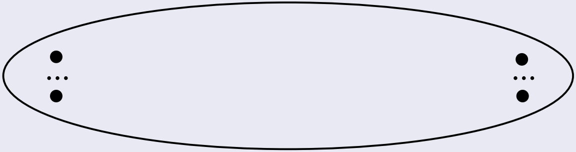
C_5-colourability is NP-complete



Gadgets

Nešetřil - Siggers 2007

A general gadget for H

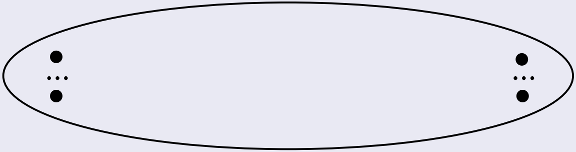


- Any H-colouring assigns different possible patterns of colours
- Any assignment of different possible patterns of colours extends to the whole gadget

Gadgets

Nešetřil - Siggers 2007

A general gadget for *H*



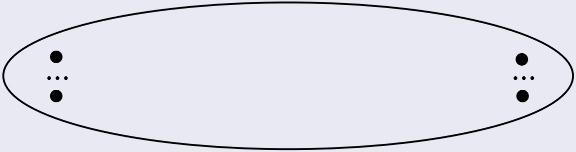
- Any *H*-colouring assigns different possible patterns of colours
- Any assignment of different possible patterns of colours extends to the whole gadget

Groups of patterns, three groups suffice

Gadgets

Nešetřil - Siggers 2007

A general gadget for H



- Any H-colouring assigns different possible patterns of colours
- Any assignment of different possible patterns of colours extends to the whole gadget

Groups of patterns, three groups suffice

Example: $\{\underline{ab}, ba\}, \{\underline{ac}, ca\}, \{\underline{bc}, cb\}$

Gadgets

Nešetřil-Siggers 2007

If H admits a gadget then H-colourability is NP-complete

Gadgets

Nešetřil-Siggers 2007

If H admits a gadget then H-colourability is NP-complete

Example $H = C_{2k+1}$

C_{2k+1}-colourability is NP-complete

Dichotomy Conjectures

Feder-Vardi Conjecture 1993

If H-colourability is not NP-complete, then it is in P

Dichotomy Conjectures

Feder-Vardi Conjecture 1993

If H-colourability is not NP-complete, then it is in P

Nešetřil-Siggers Conjecture 2007

If H does not admits a gadget then H-colourability is polynomial

Dichotomy Conjectures

Equivalent conjectures

- If the algebra of polymorphisms associated with H does not admit a non-trivial divisor in which all term operations are projections, then H-colourability is polynomial (Bulatov-Jeavons 2001)
- If H admits a Taylor polymorphism then H-colourability is polynomial (Larose-Zádori 2004)
- If H admits a weak near-unanimity polymorphism then H-colourability is polynomial (Maróti-McKenzie 2006)
- If H has an asymptotically resilient polymorphism then H-colourability is polynomial (Kun-Szegedy 2008)
- If H admits a cyclic polymorphism then H-colourability is polynomial (Barto-Kozik 2012)

Dichotomy Conjectures

Equivalent conjectures

- If the algebra of polymorphisms associated with H does not admit a non-trivial divisor in which all term operations are projections, then H-colourability is polynomial (Bulatov-Jeavons 2001)
- If H admits a Taylor polymorphism then H-colourability is polynomial (Larose-Zádori 2004)
- If H admits a weak near-unanimity polymorphism then H-colourability is polynomial (Mároti-McKenzie 2006)
- If H has an asymptotically resilient polymorphism then H-colourability is polynomial (Kun-Szegedy 2008)
- If H admits a cyclic polymorphism then H-colourability is polynomial (Barto-Kozik 2012)

Survey H-Nešetřil 2008

How to use the absence?

Nesetril-Siggers Conjecture 2007

If H does not admits a gadget then H-colourability is polynomial

How to use the absence?

Nesetril-Siggers Conjecture 2007

If H does not admits a gadget then H-colourability is polynomial

Nesetril-H 1990: Undirected graphs without loops

If H contains an odd cycle, then H-colourability is NP-complete
Otherwise H-colourability is polynomial

How to use the absence?

Nesetril-Siggers Conjecture 2007

If H does not admits a gadget then H-colourability is polynomial

Nesetril-H 1990: Undirected graphs without loops

If H contains an odd cycle, then H-colourability is NP-complete
Otherwise H-colourability is polynomial

Easy if H is bipartite

How to use the absence?

Nesetril-Siggers Conjecture 2007

If H does not admits a gadget then H-colourability is polynomial

Nesetril-H 1990: Undirected graphs without loops

If H contains an odd cycle, then H-colourability is NP-complete
Otherwise H-colourability is polynomial

Easy if H is bipartite

Gadget is simple for an odd cycle, but not simple for an H containing an odd cycle

How to use absence?

Suppose H is an (induced) subgraph of H'

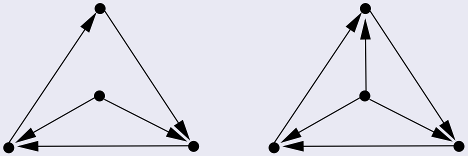
Does H-colourability reduce to H'-colourability?

How to use absence?

Suppose H is an (induced) subgraph of H'

Does H-colourability reduce to H'-colourability?

For digraphs



Situations with monotonicity

Situations with monotonicity

List H-colouring

Each vertex x of the input digraph G has a *list* $L(x) \subseteq V(H)$

Situations with monotonicity

List H-colouring

Each vertex x of the input digraph G has a *list* $L(x) \subseteq V(H)$
Is there an H-colouring f of G for which all $f(x) \in L(x)$?

Situations with monotonicity

List H-colouring

Each vertex x of the input digraph G has a *list* $L(x) \subseteq V(H)$
Is there an H-colouring f of G for which all $f(x) \in L(x)$?

Suppose H is an induced subgraph of H'

Then list-H-colourability reduces to list-H'-colourability

An instance of list-H-colourability is also an instance of list-H'-colourability

Situations with monotonicity

Minimum cost H

Input is a digraph G, with costs $c(x, y)$ for each pair $x \in V(G), y \in V(H)$, and a target total cost C

Situations with monotonicity

Minimum cost H

Input is a digraph G, with costs $c(x, y)$ for each pair $x \in V(G), y \in V(H)$, and a target total cost C

Is there an H-colouring f of G with $\sum_{x \in V(G)} c(x, f(x)) \leq C$?

Situations with monotonicity

Minimum cost H

Input is a digraph G, with costs $c(x, y)$ for each pair $x \in V(G), y \in V(H)$, and a target total cost C

Is there an H-colouring f of G with $\sum_{x \in V(G)} c(x, f(x)) \leq C$?

Suppose H is an induced subgraph of H'

Now mincost-H-colourability reduces to mincost-H'-colourability

An instance of mincost-H-colourability extends to an instance of mincost-H'-colourability

Reflexive Undirected Graphs

(Every vertex has a loop)

Reflexive Undirected Graphs

(Every vertex has a loop)

H-colourability is trivial,

but list-H-colourability

and mincost-H-colourability are not

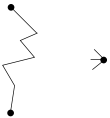
Reflexive Undirected Graphs

Feder-H 1998

If H contains an induced cycle > 3 or asteroidal triple (AT), then list-H-colourability is NP-complete
Otherwise list-H-colourability is polynomial

AT = Asteroidal Triple

u, v, w, any pair joined by a path avoiding the neighbourhood of the third



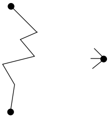
Reflexive Undirected Graphs

Feder-H 1998

If H contains an induced cycle > 3 or asteroidal triple (AT), then list-H-colourability is NP-complete
Otherwise list-H-colourability is polynomial

AT = Asteroidal Triple

u, v, w, any pair joined by a path avoiding the neighbourhood of the third



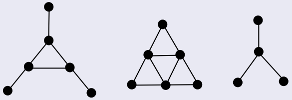
Lekkerkerker-Boland 1962

H contains no an induced cycle > 3 and no AT $\Longleftrightarrow$
H is an interval graph

Reflexive Undirected Graphs

Gutin-H-Rafiey-Yeo 2008

If H contains an induced cycle > 3, claw, net, or tent, then mincost-H-colourability is NP-complete

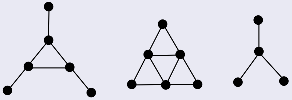


Otherwise mincost-H-colourability is polynomial

Reflexive Undirected Graphs

Gutin-H-Rafiey-Yeo 2008

If H contains an induced cycle > 3, claw, net, or tent, then mincost-H-colourability is NP-complete



Otherwise mincost-H-colourability is polynomial

Wegner 1967

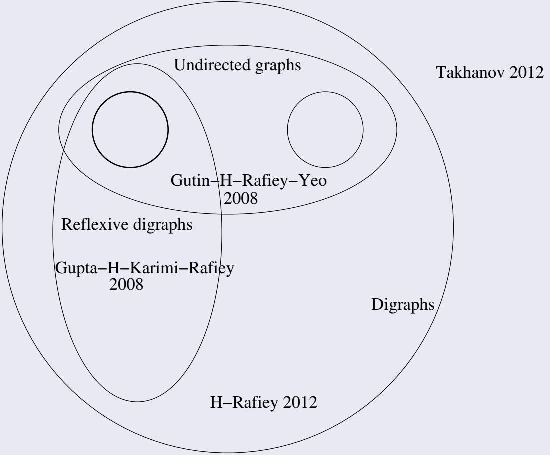
H contains no induced cycle > 3, claw, net, or tent
$\iff$ H is a proper interval graph

Mincost-H-colourability

Gradual progress

Mincost-H-colourability

Gradual progress



Mincost-H-colourability

Suppose H has no unbalanced cycles

Mincost-H-colourability

Suppose H has no unbalanced cycles

H-Rafiey 2012

If H contains a symmetrically invertible pair, then mincost-H-colourability is NP-complete
Otherwise list-H-colourability is polynomial

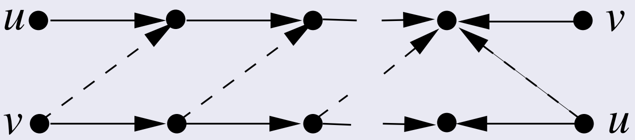
Mincost-H-colourability

Suppose H has no unbalanced cycles

H-Rafiey 2012

If H contains a symmetrically invertible pair, then mincost-H-colourability is NP-complete
Otherwise list-H-colourability is polynomial

Symmetrically invertible pair u, v



Mincost-H-colourability

Suppose H has no unbalanced cycles

H-Rafiey 2012

If H contains a symmetrically invertible pair, then mincost-H-colourability is NP-complete
Otherwise list-H-colourability is polynomial

H-Rafiey 2012

H does not contain a symmetrically invertible pair
$\Longleftrightarrow$ it is a monotone proper interval digraph

Mincost-H-colourability

Suppose H has unbalanced cycles

H-Rafiey 2012

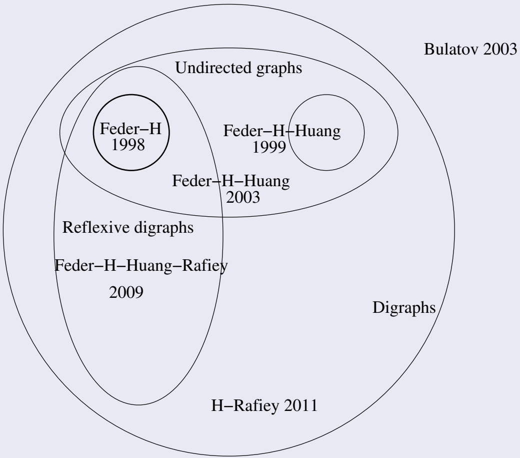
If H contains two induced unbalanced cycles of different net lengths, then mincost-H-colourability is NP-complete

Otherwise H admits a homomorphism to a directed cycle, and if H contains a symmetrically invertible pair the same circular level, then mincost-H-colourability is NP-complete

In all other cases mincost-H-colourability is polynomial

List-H-colourability

Gradual progress



List-H-colourability

H-Rafiey 2011

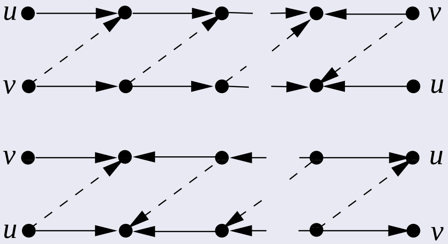
If H contains a digraph asteroidal triple (DAT), then list-H-colourability is NP-complete
Otherwise list-H-colourability is polynomial

List-H-colourability

H-Rafiey 2011

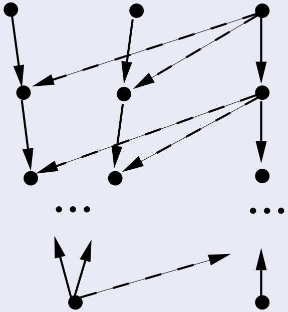
If H contains a digraph asteroidal triple (DAT), then list-H-colourability is NP-complete
Otherwise list-H-colourability is polynomial

Invertible pair u, v



List-*H*-colourability

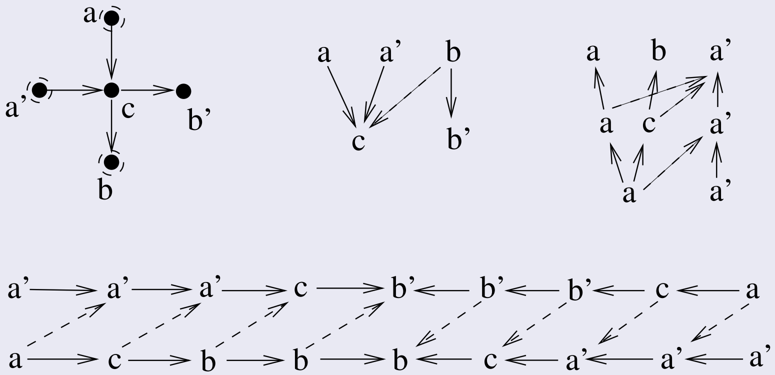
What is a digraph asteroidal triple?



invertible pair

List-H-colourability

Example DAT



List-H-colourability

How to use the absence of DATs?

List-H-colourability is in P if H admits for every pair of vertices a polymorphism that is locally semi-lattice, majority, or Maltsev on that pair
Otherwise it is NP-complete

Bulatov 2003

Dichotomy Conjectures

Larose-Tesson Conjecture 2007

If list-H-colourability is not NL-hard, then it is in L

Dichotomy Conjectures

Larose-Tesson Conjecture 2007

If list-H-colourability is not NL-hard, then it is in L

Larose-Tesson Conjecture 2007

If H admits a sequence of Hagemann-Mitschke polymorphisms then list-H-colourability is in L
Otherwise list-H-colourability is NL-hard

Reflexive Undirected Graphs

Egri-Krokhin-Larose-Tesson 2012

If H contains an induced P_4 or C_4, then list-H-colourability is NL-hard

Otherwise list-H-colourability is in L

Reflexive Undirected Graphs

Egri-Krokhin-Larose-Tesson 2012

If H contains an induced P_4 or C_4, then list-H-colourability is NL-hard
Otherwise list-H-colourability is in L

Trivially perfect graphs

H is trivially perfect if and only if it does not contain an induced P_4 or C_4

Reflexive Undirected Graphs

Egri-Krokhin-Larose-Tesson 2012

If H contains an induced P_4 or C_4, then list-H-colourability is NL-hard
Otherwise list-H-colourability is in L

Trivially perfect graphs

H is trivially perfect if and only if it does not contain an induced P_4 or C_4

Intersection of the classes of interval graphs and cographs

Reflexive Undirected Graphs

Egri-Krokhin-Larose-Tesson 2012

If H contains an induced P_4 or C_4, then list-H-colourability is NL-hard
Otherwise list-H-colourability is in L

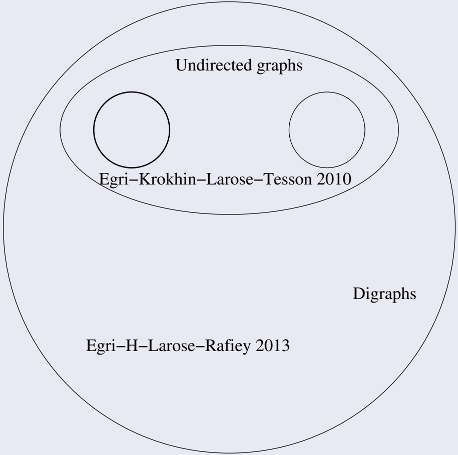
Trivially perfect graphs

H is trivially perfect if and only if it does not contain an induced P_4 or C_4

Intersection of the classes of interval graphs and cographs
Trivially perfect graphs admit a simple recursive definition (Yan-Chen-Chang 1996)

List-H-colourability in L

Gradual progress



List-H-colourability

Egri-H-Larose-Rafiey 2013

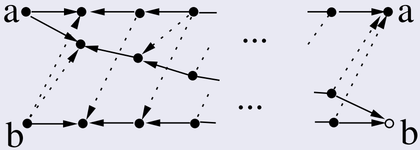
If H contains a circular N, then list-H-colourability is NL-hard
Otherwise H admits a sequence of Hagemann-Mitschke polymorphisms and list-H-colourability is polynomial

List-H-colourability

Egri-H-Larose-Rafiey 2013

If H contains a circular N, then list-H-colourability is NL-hard
Otherwise H admits a sequence of Hagemann-Mitschke polymorphisms and list-H-colourability is polynomial

What is a circular N?

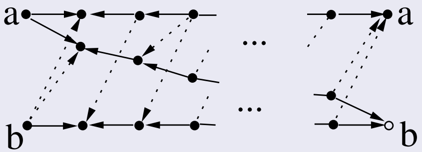


List-H-colourability

Egri-H-Larose-Rafiey 2013

If H contains a circular N, then list-H-colourability is NL-hard
Otherwise H admits a sequence of Hagemann-Mitschke polymorphisms and list-H-colourability is polynomial

What is a circular N?



How to use the absence of circular N's?

A recursive algorithm using Reingold 2005

A detailed classification

A detailed classification

Let H be a fixed digraph

A detailed classification

Let H be a fixed digraph

- If H contains a DAT, then list-H-colourability is NP-complete

A detailed classification

Let H be a fixed digraph

- If H contains a DAT, then list-H-colourability is NP-complete
- If H is DAT-free but contains a circular N, then list-H-colourability is polynomial time solvable, but NL-hard

A detailed classification

Let H be a fixed digraph

- If H contains a DAT, then list-H-colourability is NP-complete
- If H is DAT-free but contains a circular N, then list-H-colourability is polynomial time solvable, but NL-hard
- If H contains no circular N, then the problem is in L

Egri-H-Larose-Rafiey 2013

The meta-problems

In terms of the size of H

- Testing the existence of a DAT is polynomial

The meta-problems

In terms of the size of H

- Testing the existence of a DAT is polynomial
- Testing the existence of a circular N is polynomial