# Problems and memories

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# First encounter - Mátraháza, 1962

- My first encounter with Paul Erdős took place in 1962 at the Mátraháza Guest House of the Hungarian Academy of Sciences.
- I was a high school student and was rather embarrassed by the solemn formalities, especially at the dinner tables.
- I was pleased to hear the signs of some unaccepted behavior from a neighboring table, where an „old" man about fifty was regulated by his mother:
- „Pali you should keep your fork properly!"

# Pingpong

- I soon learned that the unruled boy is a famous mathematician who travels around the world with his mother.
- Next day I had opportunity to play ping pong against „Pali"
- I was very angry upon being beaten by such an old man playing in a ridiculous style.
- As a consolation he told me what a graph is and what does Turán theorem say about the number of edges in a graph that does not contain $K_{k+1}$. „Adding any edge to the Turán graph we get a $K_{k+1}$ but...
- ...what is the smallest graph with this property?" - asked during the revenge game which I lost again.

# Budapest, 1963

- The problem I have heard from Erdős in Mátraháza kept me busy and about a year later, after a lecture he gave in Budapest, I handed him my typewritten solution for the case when $n$, the number of vertices, is large in terms of $k$.
- It was disappointing to learn that his question was not an unsolved problem, but his result with Hajnal and Moon!

**Theorem**

*(Erdős, Hajnal, Moon, 1964) The minimum number of edges in a $K_{k+2}$-saturated graph with $n$ vertices is $\binom{k}{2} + k(n-k)$.*

# Eötvös University, 1965

- Two years later, at Eötvös University, I listened to Béla Bollobás' talk at the Hajós seminar about extending the Erdős - Hajnal - Moon result to hypergraphs.
- I remember Béla's comment: „this is trivial"...
- Trivial or not, certainly important and rediscovered several times (Jaeger - Payan 1971, Katona 1974). The underlying idea, cross-intersecting sets, developed further and have many applications.

# Memphis, Tennessee, 1980 - 1992

- Long distance information, give me Memphis Tennessee,
- Help me find the party trying to get in touch with me,
- She could not leave her number, but I know who placed the call,
- 'Cause my uncle took the message and he wrote it on the wall...
- ... and beside Elvis Presley, Chuck Berry and Fedex, I learned that Memphis Tennessee is also known as a hub during the movements of Paul Erdős in the US.
- Indeed, I have had much more chance to meet (and think on math) with him there than in Budapest.
- The University of Memphis (with Faudree, Ordman, Rousseau and Schelp as permanent faculty and me as permanent visitor) provided many opportunities to pursue theorems, problems and conjectures.
- From the early 90-s the department was fortified by Béla Bollobás, who leads a Chair of Excellence in Combinatorics since then.

**1. Cyles in graphs without proper subgraphs of minimum degree 3.**

- Graphs with $n$ vertices and $2n-1$ edges must contain proper subgraphs of minimum degree 3 but this fails for graphs with $n$ vertices and $2n-2$ edges, for example the wheel is such a graph (Erdős, Faudree, Rousseau and Schelp, 1990).

Let $G(n)$ be the family of graphs with $n$ vertices, $2n-2$ edges and without proper subgraphs of minimum degree 3.

**Conjecture**

**1.** (Erdős, Faudree, Gy., Schelp, 1988) Every $G \in G(n)$ contains cycles of length $i$ for every integer $3 \leq i \leq k$ where $k$ tends to infinity with $n$.

**2. Large chordal subgraphs.**

- Any graph with $n$ vertices and at least $n^2/3$ edges contains a chordal subgraph with at least $2n-3$ edges. The complete tripartite graph shows that this is sharp

### Conjecture

**2.** (Erdős , Gy., Ordman, Zalcstein 1989) Any graph with $n$ vertices and more than $n^2/3$ edges contains a chordal subgraph with at least $8n/3-4$ edges. The complete tripartite graph with one additional edge shows that this would be sharp.

We could prove a weaker result, that graphs with $n$ vertices and more than $n^2/3$ edges contain chordal subgraphs with at least $7n/3-6$ edges.

**3. Monochromatic domination.**

- In an edge colored complete graph $K$ a subset $S$ of vertices dominate in color $i$ those vertices in $V(K) - S$ that send at least one edge of color $i$ to $S$.
- If the edges of $K_n$ are 3-colored then in one of the colors at most 22 vertices dominate at least $2n/3$ vertices of $K_n$.
- However, there are 3-colorings such that no color can dominate more than $2n/3$ vertices of $K_n$ with any fixed number of vertices.
- The random 3-coloring shows that no two vertices dominate much more than $5n/8$ vertices in any color.

**Problem**

**3.** (Erdős, Faudree, Gould, Gy., Rousseau, Schelp, 1990) If the edges of $K_n$ are 3-colored then in one of the colors at most 3 vertices dominate at least $2n/3$ vertices of $K_n$. Recently Kral, Liu, Sereni, Whalen and Yilma applied flag algebra to prove that in one of the colors 4 vertices dominate at least $2n/3$ vertices of $K_n$.

### 4. Covering by monochromatic cycles

- The cycle partition number be the minimum $k$ such that the vertex set of any $r$- edge-colored complete graph can be covered by at most $k$ vertex disjoint monochromatic cycles.
- The cycle partition number depends only on $r$ and it is at most $cr^2 \log r$ (Erdős, Gy. - Pyber, 1991)

**Conjecture**

**4.** (Erdős, Gy., Pyber, 1991) The cycle partition number of any $r$-colored complete graph is at most $r$.

The case $r = 2$ in Conjecture is due to J. Lehel and was proved for large enough complete graphs by Łuczak, Rödl and Szemerédi and Allen. Then Bessy and Thomassé proved it for all complete graphs. Although Conjecture 4 for $r = 3$ is asymptotically true, Pokrovskiy found a counterexample in which three monochromatic cycles cannot cover all vertices.

## 5. Bipartite graphs plus a matching

- At a conference in Orsay in 1976 Paul and me talked about 4-critical (4-chromatic but removing any edge becomes 3-colorable) graphs that can be written as the union of a bipartite graph and a matching. We returned to this problem in Memphis calling them 4-critical $B+M$-graphs. A $B+3$ graph is a graph which can be written as the union of a bipartite graph and a matching with three edges.

### Problem

**5.** (Chen - Erdős, Gy. - Schelp, 1997) „We know that one has to be careful with conjectures in this area. That is why we only suspect that 4-critical $B+3$-graphs on $n$ vertices must have at least $2n$ edges asymptotically and dare to conjecture only that they have significantly more than $5n/3$ edges."

# Szentendre 1992 - 1996

- During the years 1993 - 1996 Paul spent with us some summer weeks as our guest at Szentendre.
- Almost all essentials for his life had been present: a mathematician, a mathematician's wife who could prepare beef Stroganoff and take part in literary and theological debates
- Although at the beginning there was no phone and it was a regular program to walk to the nearest phone booth.
- Another program was to walk to vista point Kada at a hilltop nearby. Or just walk in the garden and enjoy the shade under the huge old walnut tree.
- Paul often invited us to dinner at the Merry Monks where we became regulars, one waiter have always greeted him asking „how are you and how are the prime numbers?"

# Problems from Szentendre

- Paul liked to sit at the terrace in front of the house struggling with problems after problems. Or to write a problem paper...

# Problems from Szentendre

Time to time he exclaimed: „It is very annoying that we do not see this!” Sometimes we were more successful: „This is enough for a paper, don't you think so?”

**6. Decreasing the diameter of triangle-free graphs.**

- For a triangle-free graph $G$ we defined $h_d(G)$ as the minimum number of edges to be added to $G$ to obtain a triangle-free graph of diameter at most $d$.
- Every connected triangle-free graph $G$ with $n$ vertices, $h_3(G) \leq n-1$ and $h_5(G) \leq \frac{n-1}{2}$.

The case $d = 3$ is left open, in particular we asked the following.

**Problem**

**6.** (Erdős, Gy., Ruszinkó, 1998) Is there a positive $\epsilon$ such that $h_4(G) \leq (1-\epsilon)n$ for every connected triangle-free graph $G$ with $n$ vertices?

**7. A problem on set mappings.**

- The first (out of 56) joint paper of Erdős and Hajnal defined set mappings as a function from proper subsets of $S$ to $S$ such that $f(A) \notin A$ for all proper subset $A \subset S$.
- They defined $H(n)$ as the smallest integer for which there exists a set mapping on $S$ with $|S| = n$ such that

$$\cup_{X \subseteq T} f(X) = S$$

  for every $T \subset S, |T| \geq H(n)$ and they proved that $\log_2 n < H(n)$ and conjectured that $H(n) - \log_2(n)$ tends to infinity with $n$.
- We tried (in vain) to make the following small step toward this conjecture.

**Problem**

**7.** (Erdős, Gy., 1999) Show that $H(n) > k + 1$ for $n = 2^k$.

# Problems from Szentendre

**8. Graphs in which every path induces a 3-chromatic subgraph.**

- During the summer of 1995 Paul cited the following from one of his problem books.
- „I asked this with Hajnal: if each odd cycle of a graph induces a subgraph with chromatic number at most $r$ then the chromatic number of the graph is bounded by a function of $r$."
- Some days later I created a warm-up to this question which is still open.

**Conjecture**

**8.** (Gy., 1997) If each path of a graph induces an at most 3-chromatic subgraph then for some constant $c$ the graph is $c$-colorable, perhaps with $c = 4$.

## 9. Balanced colorings

- We called an edge coloring of $K_n$ with $r$ colors *balanced* if every subset of $\lfloor n/r \rfloor$ vertices contains at least one edge in each color.
- Balanced $r$-coloring of $K_n$ exists when $n = r^2 + r + 1$ and $r + 1$ is a prime power.
- We conjectured that this result gives the smallest $n$ for which balanced $r$-coloring exists.

### Conjecture

**9.** (Erdős, Gy., 1999) In every $r$-coloring of the edges of $K_{r^2+1}$ there exist $r + 1$ vertices with at least one missing color among them ($r \geq 3$, true for $r = 3, 4$).

## 10. Nearly bipartite graphs.

- Erdős and Hajnal asked if every subset $S$ of vertices in a graph $G$ contains an independent set of size at least $\lfloor \frac{|S|}{2} \rfloor - k$ then can one remove $f(k)$ vertices from $G$ so that the remaining graph is bipartite? This is settled in the affirmative by Bruce Reeed.
- I looked at the case $k = 0$ and called a graph $G$ *nearly bipartite* if every $S \subseteq V(G)$ has an independent set of size $\lfloor \frac{|S|}{2} \rfloor$.
- A graph is nearly bipartite if and only if it contains neither two vertex disjoint odd cycles nor odd subdivisions of $K_4$ (where each edge is subdivided to form an odd path).

### Conjecture

**10.** (Gy. 1997) A nearly bipartite graph can be made bipartite by deleting at most 5 vertices.

Molloy and Reed believed they can prove Conjecture 10 but so far they did not work out the details...

**11. Covering by monochromatic paths.**

- I mentioned to Paul that the vertices of a 2-colored complete graph can be covered by the vertices of two monochromatic paths - an easy exercise, a footnote in my first paper with Gerencsér.
- Paul said he did not believe that and it turned out soon that he thought that the covering paths must have the same color. Within two weeks we arrived to a partial answer to this new problem.
- The vertex set of a 2-colored $K_n$ can be always covered by the vertices of at most $2\sqrt{n}$ monochromatic paths of the same color.

**Problem**

**11.** (Erdős, Gy., 1995) Is it possible to cover the vertex set of a 2-colored $K_n$ with at most $\sqrt{n}$ monochromatic paths of the same color? This would be best possible.

**12. $(p,q)$-coloring of complete graphs.**

- It is sad to look at the submission date on our paper: September 15, 1996 - one day before Paul died in Varsaw.
- We called an edge coloring of a complete graph $K_n$ a $(p,q)$-*coloring* if every $K_p \subset K_n$ spans at least $q$ colors. Thus a $(p,2)$-coloring means that there is no monochromatic $K_p$, a $(3,3)$-coloring means that the coloring is proper.
- Let $f(n,p,q)$ denote the minimum number of colors needed for a $(p,q)$-coloring of $K_n$. Some of the general bounds of this paper have been improved by Sárközy and Selkow. There are much improvements on interesting small cases as well.
- On $f(n,4,3)$ the best upper bound is $n^{o(1)}$ given in Mubayi. The lower bound of Kostochka and Mubayi is improved to $c\log n$ by Fox and Sudakov. Lower bounds of Erdős-Gy. and upper bounds of Mubayi imply that $f(n,4,4) = n^{1/2+o(1)}$.

**12. $(p, q)$-coloring - continued.**

- In the next problem the debate of the authors whether the lower or upper bound is closer to the truth is not resolved (yet).

**Problem**

**12.** (Erdős, Gy., 1997) $\frac{5(n-1)}{6} \leq f(n, 4, 5) \leq n$ - improve the estimates!

One of my favorite problems is to decide whether $f(n, 5, 9)$ is linear which is equivalent to the following:

**Problem**

**13.** (Erdős, Gy., 1997) Is there a constant $c$ such that $K_n$ has a proper edge coloring with $cn$ colors, such that the union of any two color classes has no path or cycle with four edges?

Géza Tóth proved $2n - 6 \leq f(n, 5, 9)$ and $f(n, 5, 9) \leq 2n^{1+c/\sqrt{logn}}$ is due to Axenovich.

# Problems and solutions

- With so many problems to ask, think about, share, transfer, it was unavoidable that sometimes Paul created some confusion.
- For me a remarkable adventure of this kind was my problem that Paul announced at the 18-th Southeastern Conference on Combinatorics and Graph Theory held in Boca Raton (February 23-27, 1987).
- Years went by, Paul forgot what happened and contributed the problem to himself...
- I mentioned his slip of mind at Szentendre in a summer evening of 1995.
- „It is not important who asked the problem, the important thing is that the problem is solved" - he said and I heartily agreed...

# Last slide...

- It was not always easy to live and work with Paul. But it was hard to except that we had no more summers with him.
- Chess, pingpong, card games, conjectures, proofs continued, but not the same passionate way.
- And the huge old walnut tree in the garden also died in 1996.
- Nevertheless, his results, conjectures, proofs and passionate love of mathematics are with us and carry over to the present and forthcoming generations of mathematicians.

# ...and a picture