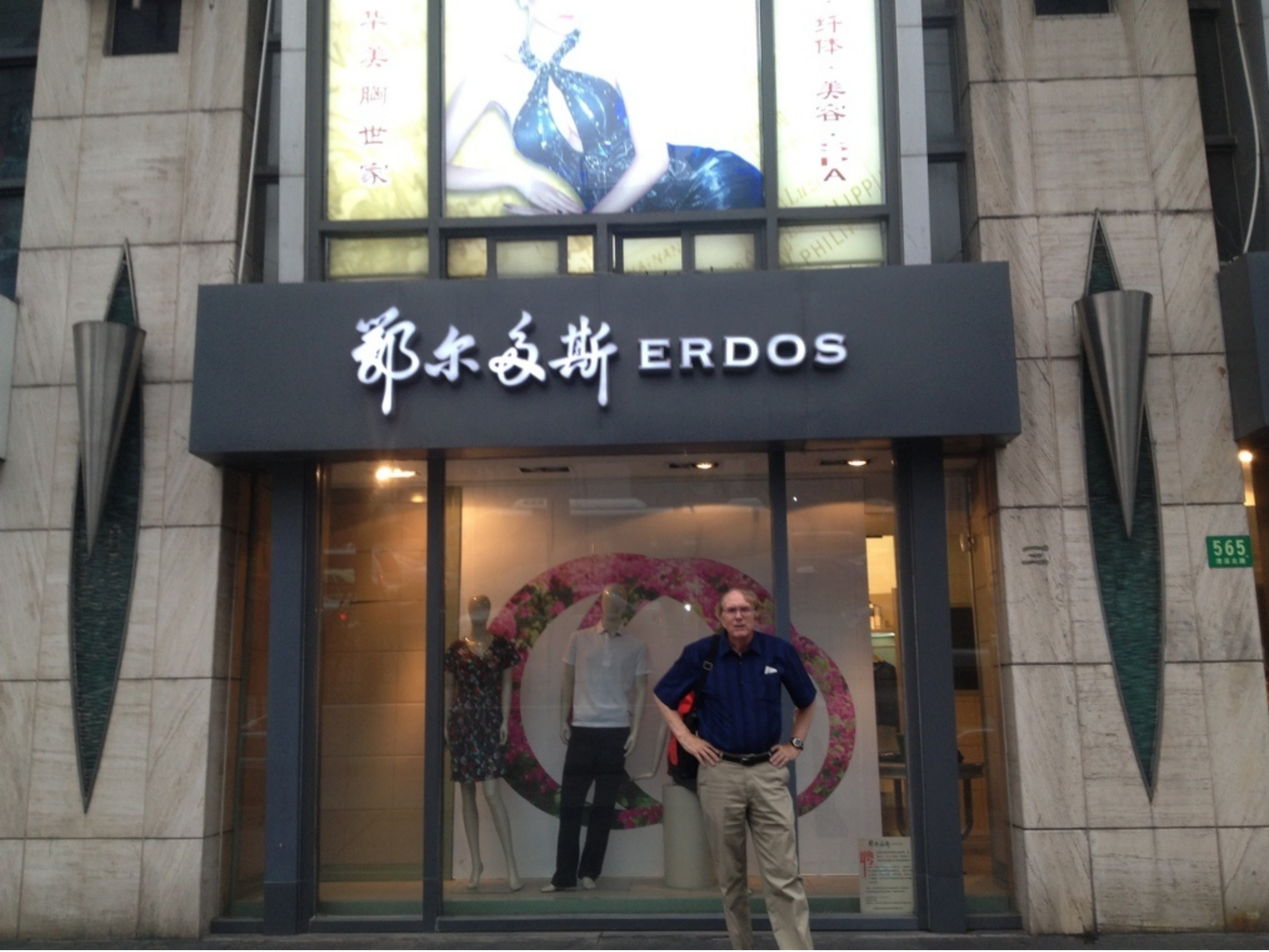
鄂尔多斯 ERDOS
565

Paul Erdös

and

Egyptian Fractions

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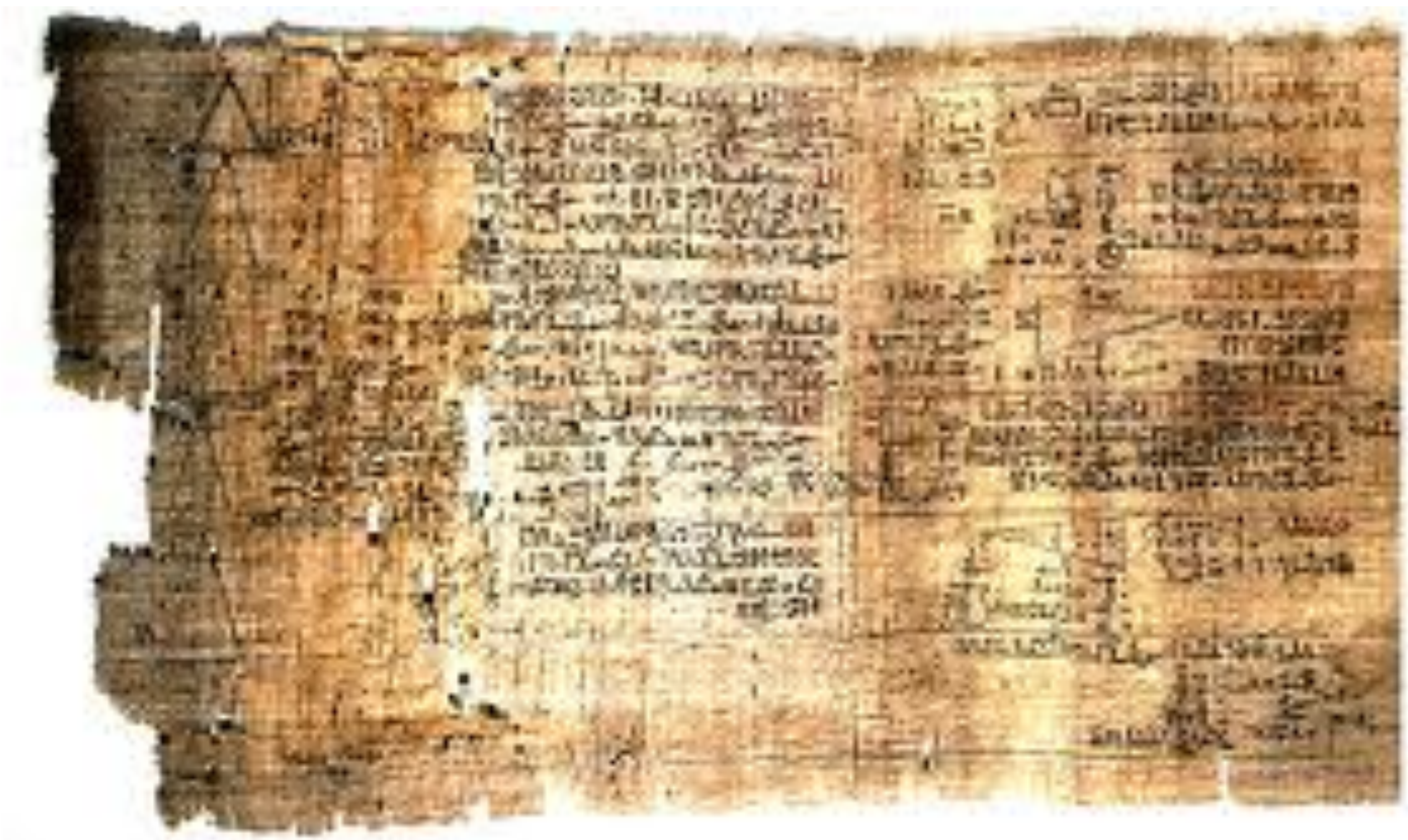

Rhind papyrus ~ 1850 B.C.

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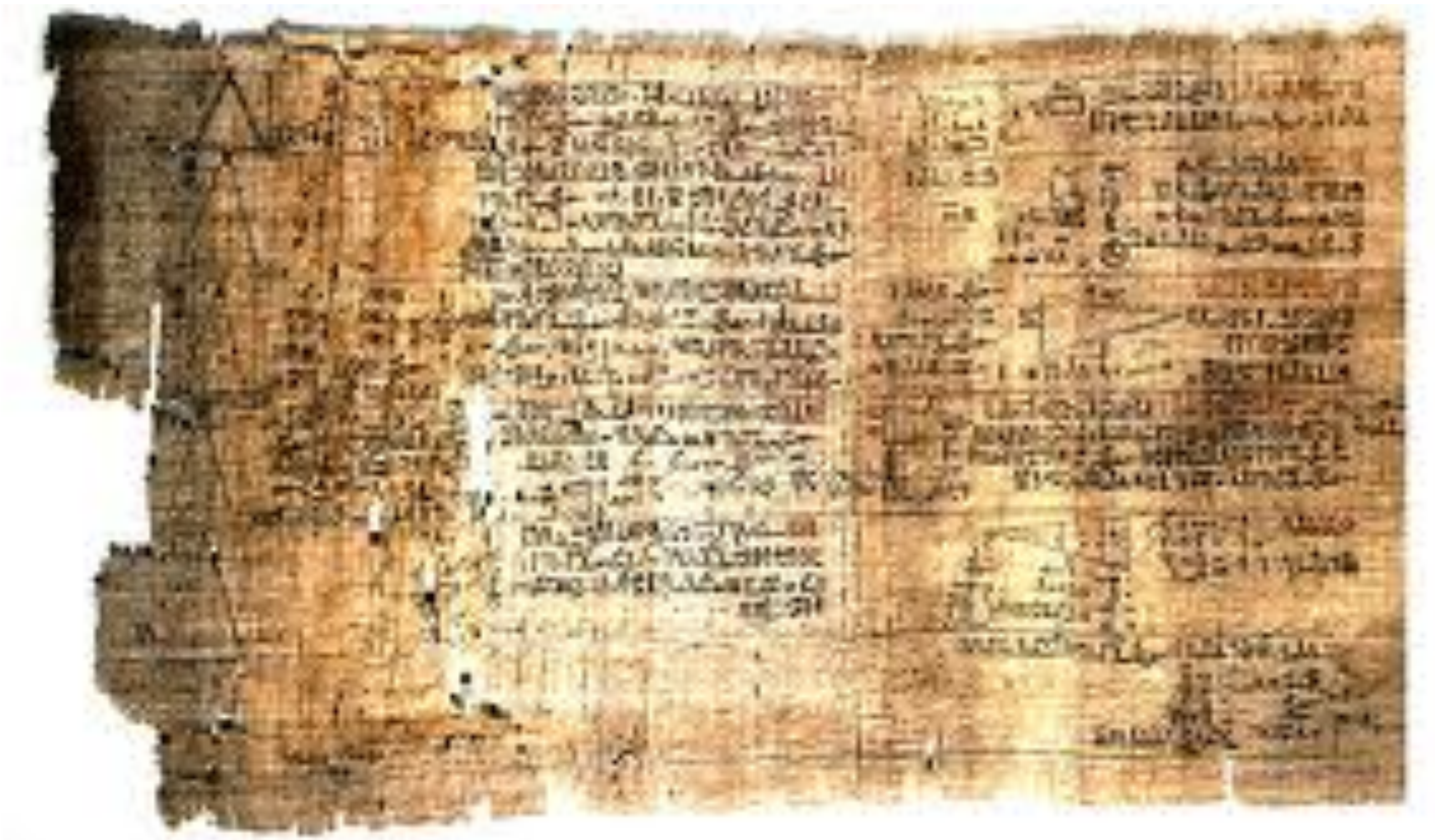

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Expressing rationals as the sum of distinct unit fractions

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Why ?

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then

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so

$$0 \le \frac{a}{b} - \frac{1}{n} = \frac{an-b}{bn}$$

and the numerator has decreased.

Another way to see this is to write

$$\frac{a}{b} = \frac{1}{b} + \frac{1}{b} + \frac{1}{b} + \ldots + \frac{1}{b}$$

and then repeatedly use the transformation

$$\frac{1}{b} = \frac{1}{b+1} + \frac{1}{b(b+1)}.$$

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Related diversion:

Show that

$$0 < x_n \leq x_{n+1} + x_{n^2} \text{ for } n \geq n_0$$

implies that $\sum_n x_n$ diverges.

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Acta Litt. Sci. Szeged **5** (1932), 194 - 198.

For every $n > 1$, there is always a prime p satisfying $n < p < 2n$.

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If $n > 2k$, then $\binom{n}{k}$ contains a prime divisor greater than k.

Erdős says: "Der Grundgedanke des Beweises besteht darin, dass ein Glied a + kd angegeben wird, welches durch eine höhere Potenz einer Primzahl teilbar ist, als die übrigen Glieder. Dies ergibt sich aus der Analyse der Primteiler der Ausdrücke: $\frac{(a+d)(a+2d)\ldots(a+nd)}{n!}$ und $\binom{2n}{n}$."

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The basic idea of the proof is that some term a + kd is divisible by a higher power of some prime than any other term. This follows from the analysis of the prime divisors of the expressions $\frac{(a+d)(a+2d)\ldots(a+nd)}{n!}$ and $\binom{2n}{n}$.

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Chen and Tang completely settled this question in 2012 by showing that the only pairs (k,n) for which the elementary symmetric function S(k,n) of $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$ is an integer are

$$S(1,1) = 1 \quad \text{and} \quad S(2,3) = 1 \cdot \tfrac{1}{2} + 1 \cdot \tfrac{1}{3} + \tfrac{1}{2} \cdot \tfrac{1}{3} = 1.$$

P. Erdős, Az $\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} = \frac{a}{b}$ egyenlet egész számú megoldásairól, Mat. Lapok **1** (1950), 192 - 210.

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Define N(a,b) to be the least value n such that the equation

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The best upper bound currently available (due to Vose) is $N(b)\ll\sqrt{\log b}$.

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What about $N(4,b)$?

The Erdős - Straus Conjecture

For all $n > 2$, $N(4,n) \leq 3$.

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The conjecture is known to hold for $n \leq 10^{14}$.

Let f(n) denote the number of solutions of

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(i) $$N \log^2 N \ll \sum_{q \le N} f(q) \ll N \log^2 N \log\log N$$

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Vaughan has shown that the number of $n \leq x$ for which the Erdos - Straus conjecture fails is $O\left(x \exp(-c(\log x)^{\frac{3}{5}})\right)$.

Sierpiński's Conjecture

For all $n \geq 5$, we can always express

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Schinzel's Conjecture

We can always express

$$\frac{a}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$$

with $0 < x_1 < x_2 < x_3$, provided $n \geq n_0(a)$.

Suppose

$$1 = \sum_{k=1}^{n} \frac{1}{x_k} \text{ where } 1 < x_1 < x_2 < \ldots < x_n.$$

Erdős conjectured that we must always have $\frac{x_n}{x_1} \geq 3$

with the extreme case being $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$.

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Note that since $H(n) = \sum_{k=1}^{n} \frac{1}{k} \sim \log n$ then we must have

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In fact, he suggested that in this case it may be that $\frac{x_n}{x_1} \to \infty$.

However, it turns out that this is not the case !

Dense unit fractions

Theorem: (Martin, Croot)

Suppose that $r > 0$ is a given rational number.

Then for all $N > 1$, there exist integers $x_1, x_2, \ldots, x_n$ with

$$N < x_1 < x_2 < \ldots < x_n \leq \left(e^r + O_r\left(\frac{\log\log N}{\log N}\right)\right)N$$

such that

$$r = \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}.$$

Moreover, the error term $O_r\left(\frac{\log\log N}{\log N}\right)$ is best possible.

Ramsey properties of Egyptian fractions

Ramsey properties of Egyptian fractions

Graham and I posed some years ago the following question: Color the integers by k colors. Is it true that

(1) $$\sum_i \frac{1}{x_i} = 1, \quad x_1 < x_2 < \dots$$

is monochromatically solvable? The sum (1) is of course supposed to be finite, but the number of summand can be as large as we please.

Let $f(m)$ be the largest integer for which there is a sequence $x_1 < x_2 < \dots x_t \leq m$, $t = f(m)$ which does not contain a solution of (1). Trivially

$$f(m) > m(1 - \frac{1}{e} - \epsilon)$$

but perhaps $f(m) = m + o(m)$. We could not get non trivial upper or lower bounds for $f(m)$.

Ramsey properties of Egyptian fractions

$500 Conjecture (Erdos - RLG)

For any r-coloring of {2,3,4,...,}, there is a monochromatic solution to

$$1 = \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}.$$

with $1 < x_1 < x_2 < \ldots < x_n$.

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(Graham) Legyen $a_1 < \dots$, $a_{k+1} - a_k < C$. Igaz-e hogy $1 = \sum \frac{1}{a_{i_r}}$ megoldható? Ugyanez kérdezhető hasonlóan azt tesszük fel, hogy $\underline{\lim}\ a_k/k < \infty$.

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For example, suppose you assume that $x_{i+1} - x_i \leq 2.$

A related result of Brown and Rödl

In any r-coloring of $\mathbb{N}$ there is always a monochromatic solution to the equation

$$\frac{1}{x} = \frac{1}{y} + \frac{1}{z}$$

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More generally, if the system $\sum_{k=1}^{n} a_{i,k} x_k = 0,\ 1 \le i \le r,$

is partition regular, then the system $\sum_{k=1}^{n} a_{i,k} x_k^{-1} = 0,\ 1 \le i \le r,$

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Warm-up: Prove this for r = 2.

Greedy decompositions in odd unit fractions

It is known that every positive rational $\frac{a}{2b+1}$ with an odd denominator can be expressed as

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Conjecture ($1000) The odd greedy algorithm always terminates.

Similar questions can be asked for more restricted unit fractions.

Theorem (RLG)

A rational $\frac{p}{q}$ can be written as the finite sum of reciprocals of distinct perfect squares if and only if:

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If $X \subset \mathbb{N}$ with $\sum_{x \in X} \frac{1}{x} = \infty$ then X contains k-AP's for every k.

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