**On the $L_\infty$-norm of the $L_2$-spline projector**

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## §1. Introduction

Many important linear operators $\mathbf{P} : X \to U$ of a linear space $X$ of functions onto a linear subspace $U$ of $X$ are defined by the minimum problem

$$\|f - \mathbf{P}f\| = \min\{\|f - u\| : u \in U\}, \quad f \in X,$$

where the semi-norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ on $X$ is induced by some semi-definite inner product $\langle \cdot, \cdot \rangle$.
Such operators $\mathbf{P}$ are **linear projectors**,
i.e. they satisfy $\mathbf{P}u = u$, for all $u \in U$.

In my talk, $X = C[a, b]$ and $\|\cdot\|_\infty$ is the uniform norm on $[a, b]$.
I am interested in the $L_\infty$-norm of the projectors $\mathbf{P}$,

$$\|\mathbf{P}\|_\infty := \sup\{\|\mathbf{P}f\|_\infty : f \in C[a, b], \quad \|f\|_\infty = 1\}.$$

The *Lebesgue function* $\Lambda_\mathbf{P} \in C[a, b]$,

$$\Lambda_\mathbf{P}(x) := \sup\{|\mathbf{P}f(x)| : \ f \in C[a, b], \ \|f\|_\infty = 1\}, \ \ x \in [a, b],$$

will play an essential role. It provides the local error estimate

$$|f(x) - \mathbf{P}f(x)| \le (1 + \Lambda_\mathbf{P}(x))\mathrm{dist}(f, U)_\infty,$$

for all $f \in C[a, b]$ and all $x \in [a, b]$.
Moreover,

$$\|\mathbf{P}\|_\infty = \max_{x \in [a,b]} \Lambda_\mathbf{P}(x)$$

and

$$\|f - \mathbf{P}f\|_\infty \le (1 + \|\mathbf{P}\|_\infty)\mathrm{dist}(f, U)_\infty,$$

## §2. Examples

**Example 1**.

Let $[a,b]=[-\pi,\pi]$, $X=\{f\in C[-\pi,\pi],\ \ f(-\pi)=f(\pi)\}$,

$$\langle f,g\rangle := \int_{-\pi}^{\pi} f(t)g(t)dt, \quad f,g\in X.$$

Let $U=\mathcal{T}_n$ be the trigonometric polynomials of degree $\leq n$,
$\mathbf{P}f=\mathbf{s}_n(f)$ is the n-th partial sum of the Fourier series of $f$.
The Lebesgue function is the constant function

$$\Lambda_{\mathbf{s}_n}(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\sin((n+1/2)t)}{\sin(t/2)}\right|dt, \quad \text{for all } x\in[-\pi,\pi],$$

and one has (L. Fejér)

$$\|\mathbf{s}_n\|_\infty=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\sin((n+1/2)t)}{\sin(t/2)}\right|dt=\frac{4}{\pi^2}\log(n+1)+O(1).$$

**Example 2**.

Let $[a,b]=[-1,1]$ and $X=C[-1,1]$.
Let $n\in\mathbb{N}$, $U=\Pi_n$ be the algebraic polynomials of degree $\leq n$.
For fixed $n+1$ points $\Delta: -1\leq t_0<t_1<\ldots<t_n\leq 1$,
let $\langle\cdot,\cdot\rangle$ be defined by $\langle f,g\rangle := \sum_{j=0}^{n} f(t_j)g(t_j)$.
The corresponding linear projector $\mathbf{P}: C[-1,1]\to\Pi_n$ is the **Lagrange interpolation operator**. Its Lebesgue function

$$\Lambda_{\mathbf{P}}(x)=\sum_{j=0}^{n}\left|\prod_{k=0,k\neq j}\frac{x-t_k}{t_j-t_k}\right|, \quad x\in[-1,1],$$

has been studied for various point sets $\Delta$ by many authors. Paul Erdös has contributed essentially to this important topic, partly in cooperation with his Hungarian colleagues Paul Turan, Jozsef Szabados, Peter Vertesi, Andras Kroo.

**Example 3**.

Let $X = C[a, b]$,
$\langle f, g\rangle = \int_a^b f(t)g(t)dt, \quad \|f\|_2 : \sqrt{\langle f, f\rangle}, \quad f, g \in X.$
Let $m \in \mathbb{N}$ and

$$\Delta : a = t_0 < t_1 < \cdots < t_n = b$$

be fixed. Let

$$\mathcal{S}_m(\Delta) := \{S \in C^{m-1}[a, b] \;\; : \;\; S_{|[t_i,t_{i+1}]} \in \Pi_m, \;\; 0 \le i \le n-1\}.$$

The $L_2$-spline projector $\mathbf{P} = \mathbf{P}_m(\Delta) : C[a, b] \to \mathcal{S}_m(\Delta)$ is defined by

$$\|f - \mathbf{P}f\|_2 = \min\{\|f - S\|_2 : S \in \mathcal{S}_m(\Delta)\}, \quad f \in C[a, b].$$

We want to study the **Lebesgue function** $\Lambda_{\mathbf{P}}$ and thus bounds for the $L_\infty$-norm Moreover,

$$\|\mathbf{P}\|_\infty = \max_{x \in [a,b]} \Lambda_{\mathbf{P}}(x)$$

of the $L_2$-spline projector $\mathbf{P} = \mathbf{P}_m(\Delta)$.

It is obvious that $\|\mathbf{P}_1(\Delta)\|_\infty \le K_1$ for some constant $K_1$ independent of $\Delta$.

Carl de Boor proved in 1968 that $\|\mathbf{P}_2(\Delta)\|_\infty \le K_2$ for some constant $K_2$ independent of $\Delta$ and formulated his famous conjecture in 1973 that

$$\|\mathbf{P}_m(\Delta)\|_\infty \le K_m \tag{1}$$

holds for all $m$ where the constant $K_m$ depend only on $m$. He proved his conjecture for $m = 3$ in 1979.

Finally, in 2001, S. Shadrin proved de Boor's conjecture for all $m \in \mathbb{N}$,

**A. Shadrin**
"The $L_\infty$-norm of the $L_2$-spline projector $\mathbf{P}_m(\Delta)$
is bounded independently of the knot sequence:
a proof of de Boor's conjecture",
Acta Math. **187** (2001), pp. 59-137.

## §3. An alternative proof of Shadrin's theorem

My new proof consists of two steps : $\mathbf{P} = \mathbf{P}_m(\Delta)$.

1. Prove that the Lebesgue function $\Lambda_{\mathbf{P}}$ is bounded at the endpoint $a$ by a constant $C_m$ depending only on $m$.
2. Prove that there exists $C_m^*$ depending only on $m$ such that

$$\max_{x\in[a,b]} \Lambda_{\mathbf{P}}(x) \le C_m^* \Lambda_{\mathbf{P}}(a)$$

   so that $\|\mathbf{P}\|_\infty \le C_m^* C_m$

It is not difficult to derive **Step 2**.
I will focuss on the proof in **Step 1**, which was much harder to find, but is easier to explain.

Let $\mathcal{B} = \{N_k\}_{k=1}^{n+m}$ be the B-spline basis of $\mathcal{S}_m(\Delta)$ for the extended knot sequence

$$\Delta_e : t_{-m} = \cdots = t_0 < t_1 < \cdots < t_n = \cdots = t_{n+m}.$$

The $L_1$-normalized basis $\{M_k\}_{k=1}^{n+m}$ of $\mathcal{S}_m(\Delta)$ is defined by

$$M_k := \frac{(m+1)N_k}{t_k - t_{k-m-1}}, \quad k = 1, \dots, n+m.$$

They have the properties (de Boor, 1976)

$$\begin{aligned}
&\mathrm{supp}(N_k) = \mathrm{supp}(M_k) = [t_{k-m-1}, t_k], \\
&N_k \ge 0, \quad M_k \ge 0, \\
&\sum_{k=1}^{n+m} N_k = 1, \quad \int_a^b M_k(t)dt = 1.
\end{aligned}$$

In Shadrin's proof, a spline $\phi \in \mathcal{S}_m(\Delta)$ plays the main role. It has the following properties :

**Theorem Φ (Shadrin [2001]).** *There exist positive numbers $c_{min}$ and $c_{max}$ depending only on $m$ such that*

$$\phi(a) = m!$$

$$sign(\langle \phi, M_j \rangle) = (-1)^{j+1}, \quad j = 1, 2, \dots, n+m \qquad (A_1)$$

$$|\langle \phi, M_j \rangle| \ge c_{min}, \quad j = 1, 2, \dots, n+m \qquad (A_2)$$

$$\|\phi\|_\infty \le c_{max} \qquad (A_3)$$

The proof of $(A1)$ and $(A2)$ is short and not difficult, while the proof of (A3) is very difficult. Fortunately, I need only $(A_1)$ and $(A_2)$ in my proof of de Boor's conjecture.

**Proof of Step 1.**

**Definition 3.1.** *We define the spline* $Q_a \in \mathcal{S}_m(\Delta)$ *by*

$$v(a) = \langle v, Q_a \rangle, \quad \textit{for all } \ v \in \mathcal{S}_m(\Delta)$$

The importance of $Q_a$ follows from

**Lemma 3.2.** *For* $\mathbf{P} = \mathbf{P}_m(\Delta)$,

$$\Lambda_{\mathbf{P}}(a) = \|Q_a\|_1 \tag{3.1}$$

**Proof**: Recall that

$$\Lambda_{\mathbf{P}}(a) := \sup\{|\mathbf{P}f(a)| : \ f \in C[a,b], \ \|f\|_\infty = 1\}.$$

Let $f \in C[a,b]$, $\|f\|_\infty = 1$.
Definition 3.1 and the orthogonality relations imply

$$\mathbf{P}f(a) = \langle \mathbf{P}f, Q_a \rangle = \langle f, Q_a \rangle.$$

Taking the supremum for $f \in C[a,b]$, $\|f\|_\infty = 1$, we obtain (3.1) for $f = \text{sign}(Q_a)$. $\square$

**Lemma 3.3.** *Let*

$$Q_a = \sum_{k=1}^{n+m} c_k M_k,$$

*then*

$$c_{k+1}c_k < 0, \quad k = 1, \ldots, n+m-1$$

Now we complete the proof of **Step 1** as follows :

Using Definition 3.1 for $v = \phi$, Lemma 3.3, $(A_1)$ and $(A_2)$, we obtain

$$\begin{aligned} m! = \phi(a) = \langle \phi, Q_a \rangle &= \sum_{k=1}^{n+m} c_k \langle \phi, M_k \rangle \\ &= \sum_{k=1}^{n+m} |c_k| |\langle \phi, M_k \rangle| \geq c_{min} \sum_{k=1}^{n+m} |c_k| \end{aligned}$$

so that

$$\|Q_a\|_1 = \Big\| \sum_{k=1}^{n+m} c_k M_k \Big\|_1 \leq \sum_{k=1}^{n+m} |c_k| \leq \frac{m!}{c_{min}} =: C_m$$

and thus by Lemma 3.2, $\Lambda_{\mathbf{P}}(a) = \|Q_a\|_1 \leq C_m$ $\square$

## §4. Local lower bounds

Recall that for $\Delta : a = t_0 < t_1 < \cdots < t_n = b$
and the $L_2$-spline projector $\mathbf{P} := \mathbf{P}_m(\Delta) : C[a,b] \to \mathcal{S}_m(\Delta)$,
the Lebesgue function $\Lambda_{\mathbf{P}} \in C[a,b]$ is defined by

$$\Lambda_{\mathbf{P}}(x) := \sup\{|\mathbf{P}f(x)| : \ f \in C[a,b], \ \|f\|_\infty = 1\}, \ x \in [a,b].$$

It satisfies ( by Shadrin's theorem )

$$\|\mathbf{P}\|_\infty = \max_{x\in[a,b]} \Lambda_{\mathbf{P}}(x) \le K_m$$

for some constant $K_m$ depending only on $m$.

Shadrin [2001] and later also his student Simon Foucart [JAT 140, 2006] proved that at the endpoint $a$

$$\sup_{\Delta} \ \Lambda_{\mathbf{P}_m(\Delta)}(a) \ \ge \ 2m+1$$

which implies that $K_m \ge 2m+1$.

Shadrin [2001] conjectures that even

$$K_m \ = \ \sup_{\Delta} \ \Lambda_{\mathbf{P}_m(\Delta)}(a) \ = \ 2m+1$$

is true.

Recently I obtained further results which seem to support Shadrin's conjecture :

**Definition.** *Let $1 \le \rho < \infty$. We say that a knot sequence $\Delta : a = t_0 < t_1 < \cdots < t_n = b$ belongs to the set $\Omega_{\rho,n}$ of knot sequences if*

$$\min_{0\le i\le n-2} \frac{t_{i+2}-t_{i+1}}{t_{i+1}-t_i} = \rho.$$

**Lemma 4.1.** *There exists a constant $\gamma_m > 0$ depending only on $m$ with the following property: If $\Delta \in \Omega_{\rho,n}$, then*

$$\left|\Lambda_{\mathbf{P}_m(\Delta)}(a) - 2m - 1\right| \le \gamma_m \left(\frac{n}{\rho} + \left(\frac{m}{m+1}\right)^n\right).$$

**Corollary 4.2.** *Let $(\rho_n)_{n=1}^\infty$ satisfy*

$$\lim_{n\to\infty} \frac{n}{\rho_n} = 0.$$

*Let $(\Delta_n)_{n=1}^\infty$ satisfy $\Delta_n \in \Omega_{\rho_n,n}$ for all $n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} \Lambda_{\mathbf{P}_m(\Delta_n)}(a) = 2m+1.$$

See also K. Höllig [JAT, 1981] for "geometric knot sequences" $\Delta_n$.

I conclude my talk with the following

**Theorem 4.3.** *For any $x \in [a,b]$, there exists a sequence $(\Delta_n)_{n=1}^{\infty}$ (depending on $x$) such that*

$$\lim_{n\to\infty} \Lambda_{\mathbf{P}_m(\Delta_n)}(x) = 2m+1.$$