Cardinals and Ordinal Definability

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Cardinals and Ordinal Definability

An old question in mathematics, dating back to Kronecker and Dedekind:

Question. To what extent can countable mathematics be done computably?

Examples:

1. A computably presentable field has a computably presentable algebraic closure.
2. A computably presentable field might not have a computable *splitting algorithm* (algorithm for testing irreducibility of polynomials defined over it).

Definability as an Analogue of Computability

In uncountable mathematics *definability* serves as an analogue of computability.

Example:
3. Any uncountable closed set of reals U is the union of a countable set C and a perfect set P (a nonempty closed set without isolated points); moreover we can *define* C and P in terms of U.
4. There are sets of reals which are not Lebesgue measurable, but in general there is no *definable* such set.

Example 3 is due to Cantor and led him to invent the theory of ordinal numbers, which in turn led him to invent *Set Theory*

So we have a new question:

Question. To what extent can Set Theory be done definably?

Ordinal-Definability

Defining something requires giving a definition and there are only countably-many definitions since there are only countably many sentences in the English (or even Hungarian) language

So set-theorists prefer to talk about definability using ordinals

Example:
Given a definable, closed set U in a topological space define:
$U_0 = U$, $U_1 = U_0'$ = all limit points of U ...
$U_{n+1} = U_n'$ ...
$U_\omega = \bigcap_n U_n$
$U_{\omega+1} = U_\omega'$ ⋯
U_α for any ordinal number
Then U_α is definable given a *name for* α, it is *ordinal-definable*, but maybe not *definable*

Gödel and Ordinal-Definability

Gödel had a lot to say about ordinal-definability

L: There is a smallest universe of set theory containing all of the ordinal numbers, the universe of *constructible sets L*
HOD: The collection of all sets which are ordinal-definable and whose elements are ordinal definable and whose elements of elements are ordinal-definable, etc. forms a universe of set theory, the universe of *hereditarily ordinal-definable sets HOD*.

L is contained in *HOD* but in general is smaller than *HOD*

Our question
Can Set Theory be done definably?
became the question
Can Set Theory be done ordinal-definably?
and this is equivalent to the question
Does V, the universe of all sets, equal HOD?

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No, V does not necessarily equal HOD.

If V equals L then Yes, but otherwise maybe not.

But modern set theory has suggested that HOD captures many features of the entire universe V, such as the notion of cardinality

HOD strongly captures cardinality iff whenever α, β are ordinals of the same cardinality then there is an ordinal-definable bijection between them.

HOD does not necessarily strongly capture cardinality: A result of Lévy shows that maybe an (infinite) ordinal is countable but there is no ordinal-definable bijection between it and ω

However we can refine this notion:

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Let κ be a cardinal number.
HOD captures cardinality κ iff whenever α, β are ordinals of cardinality κ then there is an ordinal-definable bijection between them.

HOD captures cardinality iff *HOD captures cardinality* κ for unboundedly many cardinals κ

Deep Fact of Core Model Theory. If *HOD* does not capture cardinality then there are universes of set theory with very large infinities (inaccessible, measurable, strong, Woodin and more)

This is evidence in favour of the

Cardinality Conjecture. HOD captures cardinality.

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Theorem

(Cummings-SDF-Golshani) The Cardinality Conjecture is False. Assuming the consistency of the existence of a supercompact cardinal, it is consistent that HOD does not capture cardinality κ for any infinite κ.

For the specialists in Set Theory I now give a hint of the proof:

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We want a model in which κ^+ of *HOD* has been collapsed for all infinite κ.
Start with GCH and a cardinal κ that is κ^{++++}-supercompact.
Force with κ^+-supercompact Radin forcing $\mathbb{R}_u$ using a measure sequence U of length a weak repeat point.
This forcing adds a "Radin club" C to κ and collapses α^+ to α for each α in C.
Define a "projection" map on $\mathbb{R}_u$ which discards information that would collapse α^+ for α in C.
Form a "projected forcing" $\mathbb{R}^{\pi}_{\pi(u)}$ out of these projected conditions.
Prove the existence of a weak projection map π in the sense of Foreman-Woodin from $\mathbb{R}_u$ to $\mathbb{R}^{\pi}_{\pi(u)}$.
Use this to show that $\mathbb{R}^{\pi}_{\pi(u)}$ has the Prikry Property and therefore preserves cardinals over V.

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Now we have:

$V \subseteq V[G^\pi] \subseteq V[G]$

where $V, V[G^\pi]$ have the same cardinals and the successors of club-many cardinals less than κ have been collapsed in $V[G]$.

Next argue that the quotient of $\mathbb{R}_u$ over $\mathbb{R}^\pi_{\pi(u)}$ has enough homogeneity to ensure that the *HOD* of $V[G]$ is contained in $V[G^\pi]$.

κ stays inaccessible (even κ^{+++}-supercompact) in $V[G]$, so we can truncate to models of set theory

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$V_\kappa \subseteq V_\kappa^{V[G^\pi]} \subseteq V_\kappa^{V[G]}$

and as the *HOD* of $V_\kappa^{V[G]}$ is contained in the *HOD* of $V[G]$ which is contained in $V[G^\pi]$, we have that in $V_\kappa^{V[G]}$, α^+ of *HOD* has been collapsed for α in the Radin club C.

Finally use a homogeneous Easton product to ensure that every cardinal is of the form α or α^+ for some α in C.

Further Questions and Comments

How badly can *HOD* fail to capture cardinality?
Is it possible that every infinite cardinal is a limit cardinal of *HOD*?
Is it possible that they are all measurable in *HOD*?

And is supercompactness necessary? Perhaps the Core Model Theory breaks down far below a supercompact.

More comments: We have focused on Definability.
Definability is not the only analogue of computability in Set Theory. Indeed there is interesting work on generalisations of the notion of computability itself in Set Theory, even with corresponding notions of computational complexity. For these theories, Gödel's model L of constructible sets seems to be sufficient and for example questions about "computable" yet uncountable fields in L are quite interesting.

Further Questions and Comments

And finally it is worth noting that there is a sense in which *HOD indeed is* a good approximation to the universe of sets as a whole:

Theorem

(a) (Vopenka) Every set is generic over HOD.
(b) The entire universe is generic over HOD.

Here the word "generic" refers to generalisations of Cohen's method of forcing to produce "generic" extensions of a given model of set theory. However by my result with Cummings and Golshani, it seems that forcing can do violent things to a model of set theory, such as ruining its cardinal structure. Nonetheless there is some evidence that there are limits to the damage that forcing can do ... but this is the subject of another talk!

Thanks for listening!