A relative Szemerédi theorem

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Introduction

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If $S \subset [N]$ satisfies certain conditions, then

any subset of S with no k-term AP has size $o(|S|)$.

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Relative Szemerédi Theorem

If $S \subset [N]$ satisfies certain conditions, then

any subset of S with no k-term AP has size $o(|S|)$.

Part 2: Show that the primes form a relatively dense subset of a set S that satisfies the desired conditions.

Triangle removal lemma

Triangle removal lemma (Ruzsa-Szemerédi 1976)

Every graph on n vertices with $o(n^3)$ triangles can be made triangle-free by removing $o(n^2)$ edges.

Application: Roth's theorem

Theorem (Roth)

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$i \in V_1, j \in V_2$ adjacent if $j - i \in A$

$j \in V_2, k \in V_3$ adjacent if $k - j \in A$

$i \in V_1, k \in V_3$ adjacent if $(k - i)/2 \in A$

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$(i, j, k) \in V_1 \times V_2 \times V_3$ is a triangle in G if and only if
the elements of the 3-term AP $j - i, (k - i)/2, k - j$ are in A.

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Hence, $|A| = o(N)$ or, by the triangle removal lemma, G has $\Omega(N^3)$ triangles and hence A contains a nontrivial 3-term AP.

Hypergraph Removal Lemma

Removal Lemma (Gowers, Nagle-Rödl-Schacht-Skokan)

Let H be a k-uniform hypergraph on h vertices.
Every k-uniform hypergraph on n vertices with $o(n^h)$ copies of H can be made H-free by removing $o(n^k)$ edges.

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- Implies Szemerédi's theorem.
- Solymosi showed it further implies the multidimensional generalization of Furstenberg-Katznelson.

Relative Hypergraph Removal Lemma

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Remark: In his proof that the Gaussian primes contain arbitrarily shaped constellations, Tao proved a relative hypergraph removal lemma with a stronger pseudorandomness condition.

Weighted Framework

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Example: The count of triangles when $k = 2$

The count of triangles in ν is

$$\mathbb{E}[\nu(x, y)\nu(x, z)\nu(y, z)].$$

We say that the count is correct if it is $1 + o(1)$.

Transference Lemma

Definition: Discrepancy pair

We say that (f, g) forms an ε-discrepancy pair if, for all $h : \binom{V}{k-1} \to [0, 1]$, we have

$$\left|\mathbb{E}\Big[(f(x) - g(x)) \prod_{y \in x, |y|=k-1} h(y)\Big]\right| \le \varepsilon.$$

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Transference lemma

If $(\nu, 1)$ is a $o(1)$-discrepancy pair and $0 \leq f \leq \nu$, then there is g with $0 \leq g \leq 1$ and (f, g) is a $o(1)$-discrepancy pair.

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For example, if $H = K_3$, then this means that

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Counting Lemma (Conlon-F.-Zhao)

If ν is a H-pseudorandom measure, $0 \leq f \leq \nu$, $0 \leq g \leq 1$, and (f, g) is a $o(1)$-discrepancy pair,

then the count of H in f and the count of H in g differ by $o(1)$.

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Example: The count of 3APs

The count of 3-term arithmetic progressions in ν is

$$\mathbb{E}[\nu(x)\nu(x+d)\nu(x+2d)].$$

We say the count is *correct* if it is $1 + o(1)$.

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Theorem (Conlon-F.-Zhao)

If ν is a k-pseudorandom measure, then any f with $0 \leq f \leq \nu$ and $\mathbb{E}[f(x)f(x+d)\cdots f(x+(k-1)d)] = o(1)$ satisfies $\mathbb{E}[f] = o(1)$.

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ν is k-pseudorandom if it contains the correct count of certain linear forms. For example, for $k = 3$, it says that

$$\mathbb{E}\Big[\prod_{i,j\in\{0,1\}} \nu(y_i + 2z_j)\nu(-x_i + z_j)\nu(-2x_i - y_j)\Big] = 1 + o(1),$$

and the same holds if any of the twelve factors are deleted.