Multiplicative structure of integers, shifted primes and arithmetic functions

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http://www.math.uiuc.edu/∼ford/erdos100.pdf

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Probabilistic model of integers 1. Kubilius' model

For primes p, let X_p be *independent* Bernoulli random variables with

$$\mathrm{Prob}(X_p = 1) = \frac{1}{p}, \quad \mathrm{Prob}(X_p = 0) = 1 - \frac{1}{p}.$$

Each models whether a random integer is divisible by p.

Theorem (Kubilius, 1956. Universal transference principle)

For any $\varepsilon > 0$, the sequence $\{X_p : p \leqslant y^\varepsilon\}$ models "within ε" the prime factors $\leqslant y^\varepsilon$ of a random integer $\leqslant y$.

Roughly speaking, for any theorem about the sequence $\{X_p : p \leqslant y^\varepsilon\}$, the corresponding theorem about prime factors of random integers will be true with a small error term.

Example: The Erdős-Kac theorem

Recall $\mathrm{Prob}(X_p = 1) = 1/p$ and $\mathrm{Prob}(X_p = 0) = 1 - 1/p$.

Example. From $\mathbf{E}X_p = 1/p$ and $\mathbf{V}X_p = 1/p - 1/p^2$, get

$$\mathbf{E}\left(\sum_{p\leqslant y^\varepsilon} X_p\right) = \log\log y + O_\varepsilon(1), \quad \mathbf{V}\left(\sum_{p\leqslant y^\varepsilon} X_p\right) = \log\log y + O_\varepsilon(1).$$

From the Central Limit Theorem for $\sum_{p\leqslant y^\varepsilon} X_p$, get

Theorem (Erdős-Kac, 1939)

Let $\omega(n)$ be the number of distinct prime factors of n. For each real z,

$$\lim_{y\to\infty} \frac{1}{y}\#\left\{n \leqslant y : \frac{\omega(n) - \log\log y}{\sqrt{\log\log y}} \leqslant z\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}t^2}\,dt.$$

Hardy-Ramanujan: $\omega(n) \sim \log\log n$ for almost all n

Kubilius' model and random walks

Kubilius, Billingsly (1960s). Connect $\omega(n,t) = \#\{p|n : p \leqslant t\}$ to Brownian motion.

- (Erdős, 1930s–). Prime factors in any interval are Poisson. Provided $I = [\exp\exp(t), \exp\exp(u)]$ isn't too short,

$$\mathbf{P}\,(\text{random integer has } k \text{ prime factors in } I) \sim \mathrm{e}^{t-u}\frac{(u-t)^k}{k!}.$$

  Normal number of prime factors is $\sim u-t$.
- Prime divisors in disjoint intervals are independent.

Probabilistic model 2 (Galambos, Maier, DeKoninck 1970s,80s). By a theorem of Rényi, these properties characterize the Poisson process: the *sequence* of (all but the smallest and the largest) prime factors of a random integer, taken on a $\log\log -$scale, behave like a random walk with exponentially distributed steps.

Recall: X has *exponential distribution* if $\mathbf{P}(X \geqslant y) = e^{-y}$ for $y > 0$.

Random walks and "Unconventional problems"

Probabilistic model 2: The sequence of prime factors of a random integer, taken on a log log −scale, behave like a random walk with exponentially distributed steps.

Theorem (Maier, Tenenbaum (1984); was a 1948 conjecture of Erdős)

Almost all integers have two divisors d_1, d_2 satisfying $d_1 < d_2 < 2d_1$.

Multiplication table problem (Erdős, 1955). Let

$$A(N) = \#\{de : 1 \leqslant d \leqslant N, 1 \leqslant e \leqslant N\}.$$

Equivalently, count integers $\leqslant N^2$ with a divisor near N.

Easy (Erdős): $A(N) = o(N^2)$. **Proof:** For most pairs (d, e),

$$\omega(de) \approx \omega(d) + \omega(e) \approx \log\log N + \log\log N = 2\log\log(N^2) + O(1).$$

Multiplication tables, II

Improved bounds by Erdős (1960) and Tenenbaum (1984).

Theorem (KF, 2008)

$$A(N) \asymp \frac{N^2}{(\log N)^c (\log\log N)^{3/2}}, \quad c = 1 - \frac{1+\log\log 2}{\log 2} \approx 0.08607$$

Key: Fine analysis of the prime factor random walk; small deviations of the prime factor random walk lead to large discrepancies in the distribution of divisors.

Open problem. Is there an asymptotic formula?

Generalization. Find the order of

$$A_k(N_1, \ldots, N_k) = \#\{d_1 \cdots d_k : 1 \leqslant d_j \leqslant N_j \ (1 \leqslant j \leqslant k)\}.$$

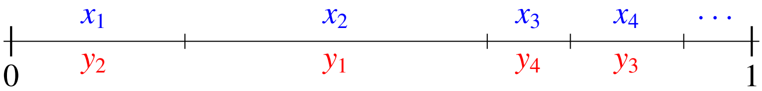
Order known for all $N_1, \ldots, N_k$ for $k = 2$ (KF, 2008), $3 \leqslant k \leqslant 6$ (Koukoulopoulos 2010, 2013). Partial results for $k > 6$.

Distribution of large prime factors

Notation: $P_1(n)$ = largest prime factor of n, $P_2(n)$ = 2nd largest, etc.

Distribution of $P_1(n)$. Early work of Ramanujan, Dickman, Erdős and others. $\Psi(x, y) = \#\{n \leqslant x : P_1(n) \leqslant y\}$ is well understood now.

Joint distribution of $P_1(n), \ldots, P_k(n)$. (Billingsly, 1972).
(Donnelly and Grimmett, 1993): It's the Poisson-Dirichlet distribution
Simple description: Let $(x_1, x_2, \ldots)$ be a random partition of $[0, 1]$:



Let y_1 = largest x_i, y_2 = the 2nd largest, etc.
Then $(y_1, y_2, \ldots)$ and $\left(\frac{\log P_1(n)}{\log n}, \frac{\log P_2(n)}{\log n}, \ldots\right)$ have the same distribution.

Same distribution appears in the cycle lengths of random permutations, factor sizes of random polynomials in $\mathbb{F}_q[t]$, certain physical processes, etc.

Anatomy of shifted primes

Sets $\mathscr{P}_a = \{p + a : p \text{ prime }\}$, where $a \neq 0$ fixed.

Used to study arithmetic functions ϕ, σ, orders in $\mathbb{Z}/p\mathbb{Z}$, primality testing, factorization algorithms, cyclotomic fields, Fermat's Last Theorem, etc. Important cases $a = -1, 1$.

Small and intermediate prime factors. Essentially the same distibution as for a random integer via sieve methods, Bombieri-Vinogradov, Gallagher. Ideas originate from 1935 paper of Erdős.

- $\omega(p + a)$ has normal order $\log\log p$ (Erdős, 1935)
- $\omega(p + a)$ satisfies the same CLT as $\omega(n)$ (Halberstam, 1956).
- $\#\{d_1 d_2 \in \mathscr{P}_a : 1 \leqslant d_i \leqslant N\} \asymp \frac{A(N)}{\log N}$ (Koukoulopoulos, 2011)

Large prime factors ($> p^{1/2}$) of shifted primes largely unknown due to lack of knowledge of primes in progressions to large moduli.

Anatomy of values of arithmetic functions

Let $\mathcal{V}_f = \{f(n) : n \in \mathbb{N}\}$, $\quad V_f(x) = \#\mathcal{V}_f(x) \cap [1, x]$.

Pillai, 1929. $V_\phi(x) = o(x)$. Idea: $\omega(n) \approx \log\log x$ for most $n \leqslant x$, and $2^{\omega(n)-1} | \phi(n)$.

Erdős, 1935. $V_\phi(x) = x(\log x)^{-1+o(1)}$. Idea: $\omega(p-1) \sim \log\log p$ for most $p|n$. Hence, for typical n, $\omega(\phi(n))$ is abnormally large.

Improvements by Erdős, Erdős-Hall, Pomerance, Maier-Pomerance.

KF, 1998. exact order of $V_\phi(x)$ found:

$$V_\phi(x) \asymp \frac{x}{\log x} \exp\Big\{C_1(\log\log\log x - \log\log\log\log x)^2 \\ + C_2 \log\log\log x + C_3 \log\log\log\log x\Big\}.$$

Same order for $V_\sigma(x)$ and for the counting function of the semigroup generated by $\mathscr{P}_a$, $a \neq 0$.

Open problem. Is there an asymptotic formula?

Euler's function. More open problems

Carmichael, 1907. $\forall\, m \in \mathcal{V}_\phi$, $\phi(x) = m$ has at least 2 solutions x.

Known: such an m, if it exists, exceeds $10^{10^{10}}$ (KF, 1998).

Known: $\forall k \geqslant 2$, $\exists m$ so that $\phi(x) = m$ has exactly k sol's (KF, 1999).

Erdős. $\forall\, C > 1$, is there an $m \in \mathcal{V}_\phi$ so that $\phi(x) = m \implies x > Cm$?

KF, 1998. Is there an $m \in \mathcal{V}_\phi$ so that $\phi(x) = m \implies 6|x$?
The corresponding question with 6 replaced by 2,3,4,5,7,8 or 9 is affirmative. I think for 6, the answer is no. Perhaps for 10 also.

Erdős. Are there infinitely many n with $\phi(n) = \phi(n+1)$?
$\forall \varepsilon$, are there infinitely many n with $|\phi(n) - \phi(n+1)| < n^\varepsilon$?
Alkan-Ford-Zaharescu (2009). True with $\varepsilon = 0.84$.

Prime Chains

Definition

Let $a \prec b$ if $b \equiv 1 \pmod{a}$; that is, $a | (b-1)$.

Prime chains: $p_1 \prec p_2 \prec \cdots \prec p_k$

Example: $2 \prec 5 \prec 11 \prec 23 \prec 47 \prec 283 \prec 2432669$

Prime chain problems arise in the study of iterates of ϕ and applications thereof; value distribution of ϕ, σ, λ; primality certificates (complexity of the Pratt certificate).

Basic question. Are there arbitrarily long prime chains?
Yes - Infinitely long (Dirichlet, 1837).

Prime chains with a given starting prime

Prime chains: $p_1 \prec p_2 \prec \cdots \prec p_k, \quad p_{j+1} \equiv 1 \pmod{p_j}$ for each j.

Theorem (Ford-Konyagin-Luca, 2010)

Let $N(x;p)$ be the number of prime chains starting at a prime p and ending at a prime $\leqslant xp$. Then for every $\varepsilon > 0$, $N(x;p) \leqslant C(\varepsilon)x^{1+\varepsilon}$.

Note $N(x;p) \geqslant \pi(xp;p,1) \approx x/\log x$.

An (perhaps unexpected) application to a 1958 conjecture of Erdős.

Theorem (Ford-Luca-Pomerance, 2010)

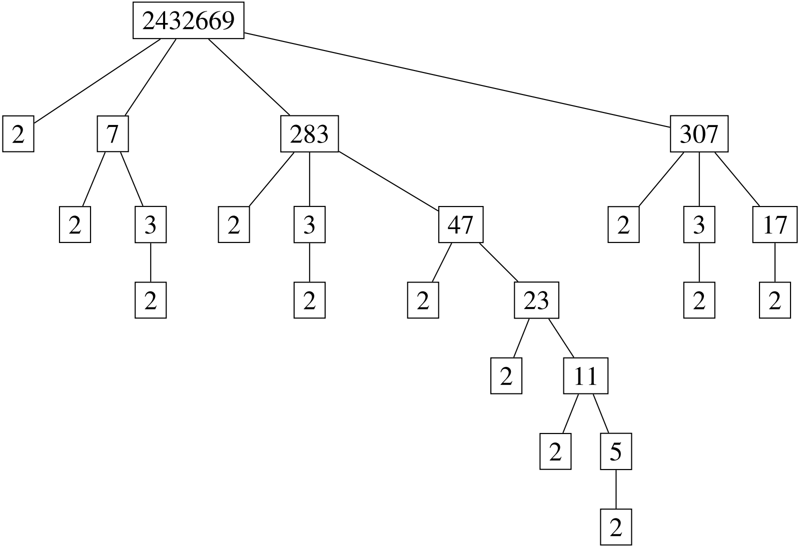
$\phi(n) = \sigma(m)$ has infinitely many solutions (i.e.,$\mathcal{V}_\phi \cap \mathcal{V}_\sigma$ is infinite)

Theorem (Ford-Pollack, 2012)

Almost all values of ϕ are not values of σ and vice-versa. That is, the counting function of $\mathcal{V}_\phi \cap \mathcal{V}_\sigma$ is $o(V_\phi(x) + V_\sigma(x))$.

Pratt trees

The aggregate of all prime chains ending at a given prime p has a *tree structure*, the **Pratt tree** of p (related to the Pratt primality certificates).



Pratt tree height

Height $H(p)$, =length of longest prime chain ending at p.
Trivially, $H(p) \leqslant \frac{\log p}{\log 2} + 1$.

$H(p) = 2$ for Fermat primes.
Conjecture (Erdős ?): For each $k \geqslant 3$, there are infinitely many primes with $H(p) = k$.

Katai, 1968. $H(p) \gg \log\log p$ for almost all p.

Ford-Konyagin-Luca, 2010. $H(p) \ll (\log p)^{0.9503}$ for almost all p.

Assuming the large prime factors of the shifted primes in the Pratt tree obey the Poisson-Dirichlet distribution, and are all independent of one another, one can model $H(p)$ be a *branching random walk*. Fine analysis of this process leads to the collowing conjecture.

Conjecture (Ford-Konyagin-Luca,2010)

For most primes p, $H(p) \approx \mathrm{e}\log\log p - \frac{3}{2}\log\log\log p + \text{“}O(1)\text{''}$.

Pratt trees with missing primes

Let $\mathscr{P}_q$ be the set of primes p such that the Pratt tree for p doesn't contain the prime q. For example,

$$\mathscr{P}_3 = \{2, 5, 11, 17, 23, 41, 47, 83, 89, 101, 137, 167, 179, 251, \ldots\}$$

Sieve methods quickly imply the counting function is $O(x/\log^2 x)$. Numerical comutations of $\mathscr{P}_3$ up to 10^{13} indicate that the counting function is $\approx x^{0.62}$.

Theorem (KF, 2013)

The counting function of $\mathscr{P}_q$ is $O(x^{1-c})$ for some positive $c = c(q)$.

Open problem. Show that $\mathscr{P}_q$ is infinite.
Likely extremely hard. $\mathscr{P}_5$ infinite (almost) implies Carmichael's conjecture.

Largest prime factors. Open problems.

Expected. $P_1(n)$ and $P_1(n+1)$ are independent.

Theorem (Erdős-Pomerance, 1978)

We have

1. $P_1(n) < P_1(n+1)$ *for a positive proportion of n;*
2. $P_1(n) > P_1(n+1)$ *for a positive proportion of n;*
3. *certain orderings of* $P_1(n-1), P_1(n), P_1(n+1)$ *occur infinitely often.*

Balog, 2001. Showed $P(n-1) > P(n) > P(n+1)$ infinitely often.

Open problem. Does any particular ordering of $P_1(n-1), P_1(n), P_1(n+1)$ occur for a positive proportion of n?

Open problem. Do all patterns (orderings) of $P_1(n), \ldots, P_1(n+3)$ occur infinitely often?

Large prime factors of shifted primes

Conjecture. $(P_1(p+a), \dots, P_k(p+a))$ has the same distribution as $(P_1(n), \dots, P_k(n))$.

True assuming Elliott-Halberstam conjecture.
Unconditionally, very little known due to lack of knowledge of primes in arithmetic progressions to large moduli.

Smooth shifted primes. Erdős (1935) showed that $P_1(p+a) < p^c$ infinitely often for some $c < 1$. **Baker-Harman, 1998**: $c = 0.2931$. Applications to ϕ and Carmichael numbers.

Large prime factors. $P_1(p+a) > p^c$ infinitely often.
Baker-Harman, 1998: $c = 0.677$.

Open problem (Buchstab). (i) Are there infinitely many primes p such that all prime factors of $p+a$ are $3 \mod 4$?
(ii) Same with $3 \mod 4$ replaced by an arbitrary $a \mod q$.

Propinquity of divisors. Hooley's Δ-function.

Theorem (Maier, Tenenbaum (1984); was a 1948 conjecture of Erdős)

Almost all integers have two divisors d_1, d_2 satisfying $d_1 < d_2 < 2d_1$.

Let $\Delta(n) = \max_y \#\{d|n : y < d \leqslant ey\}$ (a concentration function).

Normal order (Maier-Tenenbaum, 1984; 2009). For almost all n,

$$(\log n)^{c-\varepsilon} < \Delta(n) < (\log n)^{\log 2+\varepsilon}, \quad c \approx 0.33827$$

They conjecture that the lower bound is closer to the truth.

Average values (Hall-Tenenbaum (lower); Tenenbaum (upper)).

$$\log\log x \ll \frac{1}{x}\sum_{n\leqslant x} \Delta(n) \ll \exp\left\{C\sqrt{\log\log x \log\log\log x}\right\}.$$

Twisted $\Delta-$functions (Daniel; de la Bretèche-Tenenbaum):

$$\Delta_f(n) = \max_{1\leqslant y<z\leqslant ey}\left|\sum_{d|n, y<d\leqslant z} f(d)\right|, \quad f = \mu, \chi, \dots$$

Prime chains ending at a given prime

Prime chains: $p_1 \prec p_2 \prec \cdots \prec p_k, \quad p_{j+1} \equiv 1 \pmod{p_j}$ for each j.

Theorem (Ford-Konyagin-Luca,2010)

Let $f(p)$ be he number of prime chains that end at a prime p. Then

$$\frac{1}{3}\log p \leqslant f(p) \leqslant 3\log p$$

for almost all p.

$f(p)$ is also the number of nodes in the Pratt tree for p.

Open Problem. Are there infinitely many p with $f(p) = o(\log p)$?

Observations: $f(p) = 2$ for Fermat primes.
$f(p)$ is small if $p - 1$ is very smooth, e.g. $f(p) = 4$ if $p = 2^a 3^b + 1$.

Miscellaneous problems

D. H. Lehmer, 1930. Is there a *composite* n with $\phi(n)|(n-1)$?
Pomerance (1977): The counting function of such n is $O(n^{1/2}(\log n)^{O(1)})$.

Open Problem: Prove there are infinitely many chains $p_1 \prec p_2 \prec p_3$ with $\frac{p_3-1}{p_2} = \frac{p_2-1}{p_1}$ (quasi-geometric progression of primes).