COMBINATORICS SECTION

ERDŐS CENTENNIAL MEETING
BUDAPEST, HUNGARY

July 1, 2013

Saturation Numbers for Graphs

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SATURATED GRAPHS

Definition

*Given a fixed graph H, a graph G is H-**Saturated** if it contains no copy of H, but $G + e$ contains a copy of H for any edge $e \notin G$.*

SATURATED AND EXTREMAL GRAPHS

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WEAKLY SATURATED GRAPHS

Definition

$wsat(n, F) = \min\{|E(G)| : |V(G)| = n, G$ *does not have* F *as a subgraph,* but *edges in* $\overline{G}$ *can be ordered such that the addition of each edge results in a new copy of* $F\}$.

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Definition

$WSat(n, F) = \{G : |V(G)| = n, |E(G)| = wsat(n, F),$ *and* G *is* F*-weakly saturated*$\}$.

EXAMPLES

Example

The complete bipartite graph $K_{n/2,n/2}$ *is a* K_3*-saturated of order* n *that has* $n^2/4$ *edges.*

EXAMPLES

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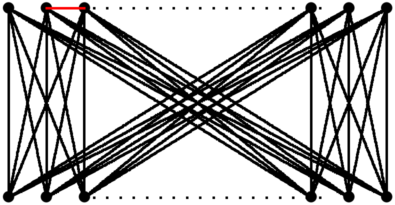
The complete bipartite graph $K_{n/2,n/2}$ is a K_3-saturated of order n that has $n^2/4$ edges.

Example

The star $K_{1,n-1}$ is a K_3-saturated of order n that has $n-1$ edges.

COMPLETE BIPARTITE – STAR GRAPHS



$K_{n/2,n/2}$

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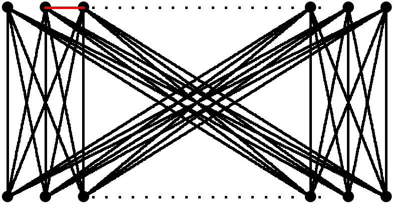

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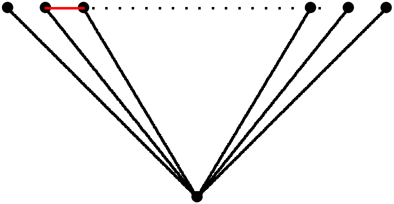
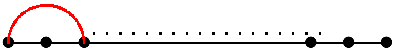


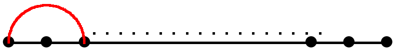
$K_{1,n-1}$

$wsat(n, K_3)$



P_n

$wsat(n, K_3)$



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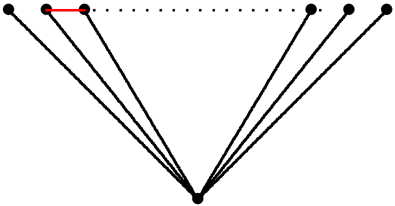

$K_{1,n-1}$

SATURATION NUMBERS FOR P_3

Theorem

$$ex(n, P_3) = \lfloor n/2 \rfloor.$$

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Theorem

$$wsat(n, P_3) = 1.$$

EXTREMAL NUMBERS FOR MATCHINGS

Theorem

For $t \geq 2$,

$$ex(n, tP_2) = (t-1)n - t(t-1)/2.$$

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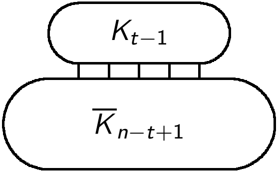
Example

The extremal graph is $K_{t-1} + \overline{K}_{n-t+1}$.



SATURATION NUMBERS FOR MATCHINGS

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$$sat(n, tP_2) = 3t - 3.$$

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Example

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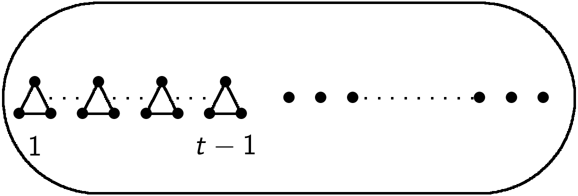

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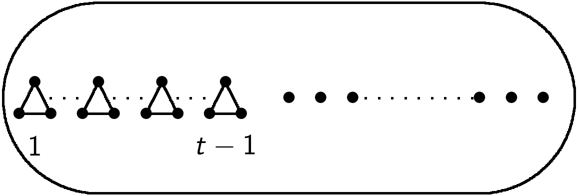

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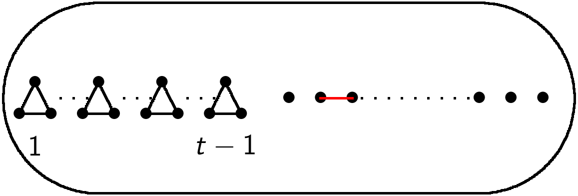

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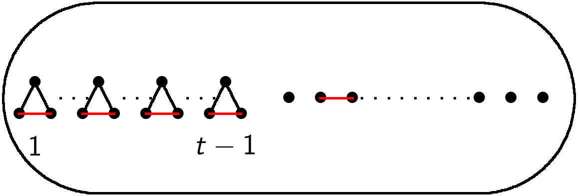

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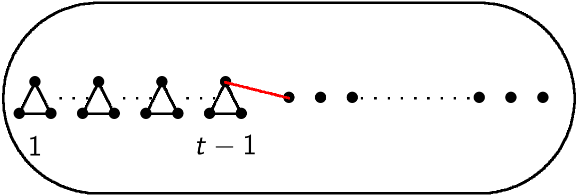

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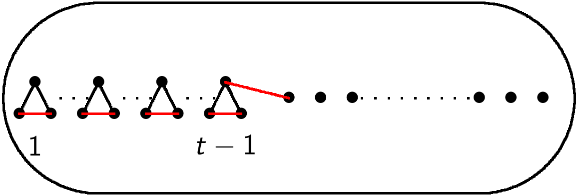

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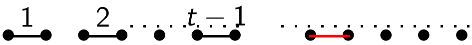

$(t-1)K_2 \cup \overline{K}_{n-2t+2}$

COMPLETE GRAPHS

Theorem

(Turán(1954)) *If* $n > t$ *and divisible by* $t - 1$*, then*

$$ex(n, K_t) = \frac{(t-2)n^2}{2(t-1)}.$$

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(Lovász (1977)) *For $n \geq t$*

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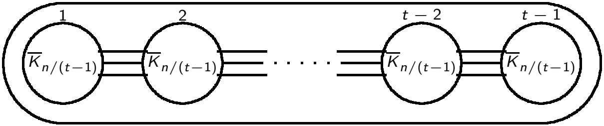
COMPLETE GRAPH EXAMPLES

Example

The extremal graph is $K_n - (t-1)K_{n/(t-1)}$.



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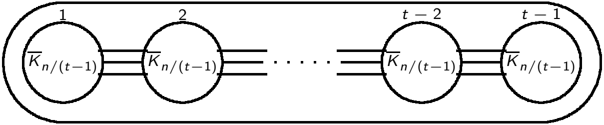
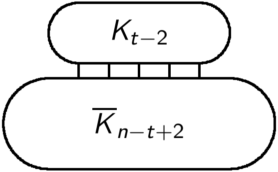


Example

The minimal saturated graph is $K_{t-2} + \overline{K}_{n-t+2}$.



GENERAL EXTREMAL THEORY

Theorem

(Erdős, Simonovits (1972)) *If n is sufficiently large and F is a graph with chromatic number $\chi(F) = p$, then*

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(Erdős, Simonovits (1972)) *For n sufficiently and F is a graph with chromatic number $\chi(F) = p$, then*

$$K_n - (p-1)\overline{K}_{n/(p-1)} \approx Ex(n, F).$$

SATURATION NUMBERS ARE LINEAR

Theorem

(Kászonyi, Tuza (1986)) *For a given graph F of order t and independence number $\alpha = \alpha(F)$, let $d = d(F)$ be the minimum degree of any vertex of $F - S$ relative to a maximum independent set S. Then,*

$$sat(n, F) \leq (t - \alpha - 1)n + \lfloor (d-1)(n - t + \alpha + 1)/2 \rfloor - \binom{t-\alpha}{2},$$

and $K_{t-\alpha-1} + H_d$, contains a saturated graph, where H_d is a $(d-1)$-regular graph of order $n - t + \alpha + 1$.

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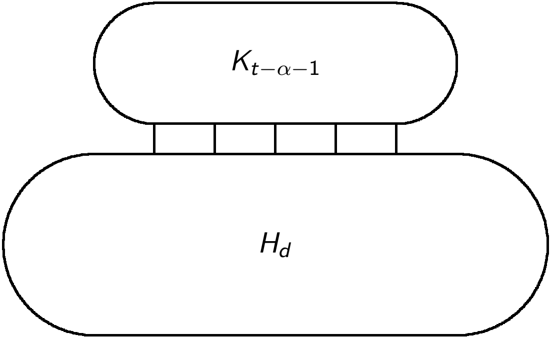
Corollary

(Kászonyi, Tuza (1986)) *For each graph F there is a constant $c = c(F)$ such that*

$$sat(n, F) < cn.$$

EXAMPLE OF EXTREMAL SATURATED GRAPH



H_d is a $(d-1)$-regular graph of order $n-t+\alpha+1$.

Theorem

Let F be a graph with p vertices, q edges, and minimal degree δ. Then, for any $n \geq p$

$$q-1+(\delta-1)(n-p)/2 \leq wsat(n,F) \leq (p-1)(p-2)/2+(\delta-1)(n-p+1).$$

GENERAL WEAK SATURATION THEORY

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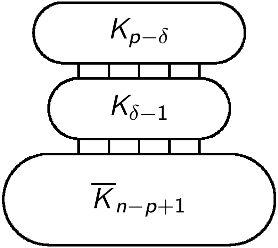

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Theorem

(F. Gould, Jacobson) *If F is a graph with p vertices and minimal degree δ, then,*

$$\frac{\delta n}{2} - \frac{n}{\delta+1} \leq wsat(n,F) \leq (\delta-1)n + (p-1)(p-2\delta)/2$$

for any n sufficiently large.

GENERAL WEAK SATURATION THEORY

Theorem

(F. Gould, Jacobson) *Let F be a graph with p vertices, q edges, and minimal degree δ. Then, for any $n \geq p$*

$$\frac{(\delta(F)-1)n}{2} + c_1 \leq wsat(n,F) \leq (\delta(F)-1)n + c_2.$$

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Theorem

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Question

Is $wsat(n,F) = (\delta(F)-1)n + c_2$ *for* $\delta(F) \geq 3$? *NO*

SATURATION NUMBER PROBLEMS

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(3) $ex(n, \mathcal{F}) \leq ex(n + 1, \mathcal{F})$

$sat(n, \mathcal{G})$ **and** $wsat(n, \mathcal{F})$ does not satisfy any of these properties.

BAD BEHAVIOR FOR (1) FOR SATURATION

Theorem

J. Faudree, R. Faudree, R. Gould, M. Jacobson *Given any positive integer C, any tree T is a subtree of a tree $T' = T'(T, C)$ such that for n sufficiently large*

$$sat(T', n) \geq Cn.$$

Any tree T' is a subtree of a tree $T" = T"(T', C)$ such that for n sufficiently large

$$sat(T", n) < n.$$

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Theorem

There are sequences of trees $T(1) \subset T(2) \subset \cdots T(m)$ such that for any positive integer C and n sufficiently large

$$sat(T(i), n) < n, \text{ for } i \text{ odd and } sat(T(i), n) > Cn, \text{ for } i \text{ even.}$$

BAD BEHAVIOR FOR (2) FOR SATURATION

Theorem

J. Faudree, R. Faudree, R. Gould, M. Jacobson *For* $t \geq 2$ *and* $n \geq t + 1$,

$$sat(n, K_{1,t} + e) = n - 1,$$

and $Sat(n, K_{1,t} + e) = \{K_{1,n-1}\}$.

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Theorem

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$$sat(n, K_{1,t} + e) = n - 1,$$

and $Sat(n, K_{1,t} + e) = \{K_{1,n-1}\}$.

Theorem

For $t \geq 2$ *and* $n \geq t+1$,

$$sat(n, \{K_{1,t} + e, K_{1,t}\}) = sat(n, K_{1,t}) = (t-1)n/2 - \frac{1}{2}\lfloor t^2/4 \rfloor.$$

BAD BEHAVIOR FOR (3) FOR SATURATION

Theorem

(Kászonyi, Tuza (1986)) *For* $t \geq 2$*,*

$$sat(2k-1, P_4) = k+1; \quad Sat(2k-1, P_4) = K_3 \cup (k-2)K_2.$$

and

$$sat(2k, P_4) = k; \quad Sat(2k, P_4) = kK_2.$$

BAD BEHAVIOR FOR (1) - WEAK SATURATION

(a) G_p graph obtained from K_p by adding a pendant edge.

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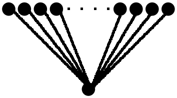
(a) G_p graph obtained from K_p by adding a pendant edge.

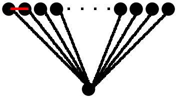
(b) $wsat(n, G_p) = \binom{p}{2}$ for all $n \geq p + 1$.

(c) $wsat(n, K_p) = \binom{p-2}{2} + (p-2)(n-p+2)$ for $n \geq p$.

BAD BEHAVIOR FOR (2) - WEAK SATURATION

(a) S_p star with p vertices, S_p^* graph obtained from S_p by adding an edge.





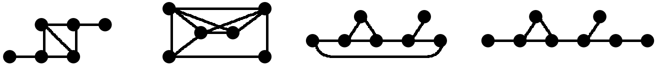
BAD BEHAVIOR FOR (2) - WEAK SATURATION

(a) S_p star with p vertices, S_p^* graph obtained from S_p by adding an edge.

(b) $wsat(n, S_p^*) = p - 1$.

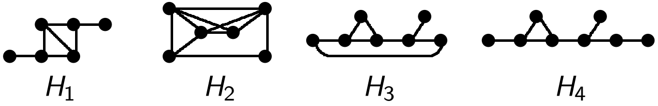
(c) $wsat(n, \{S_p, S_p^*\}) = \binom{p-1}{2}$.

BAD BEHAVIOR FOR (3) - WEAK SATURATION



H_1 H_2 H_3 H_4

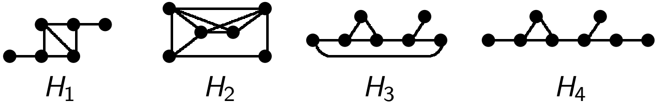
BAD BEHAVIOR FOR (3) - WEAK SATURATION



(a) $wsat(6, H_1) = 7.$

(b) $wsat(7, H_1) = 6$

BAD BEHAVIOR FOR (3) - WEAK SATURATION



(a) $wsat(6, H_1) = 7$.

(b) $wsat(7, H_1) = 6$

(a) $wsat(6, 2K_3) = 10$ and $H_2 \in WSat(6, K_3)$.

(b) $wsat(7, 2K_3) = 8$ and $H_3 \in WSat(6, K_3)$.

(c) $wsat(8, 2K_3) = 8$ and $H_4 \in WSat(6, K_3)$.

Theorem

Let F be a graph with p vertices, q edges, and minimal degree δ. Then,

$$wsat(n, F) \leq wsat(p, F) + (\delta - 1)(n - p)$$

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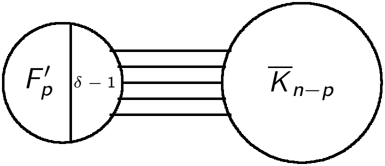

Theorem

Let F be a graph with p vertices, q edges, and minimum degree $\delta = \delta(F)$. Let $q' = wsat(p, F)$. Then, for n divisible by p,

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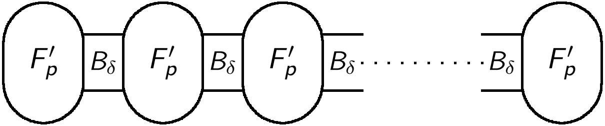

$B_5 =$ 

SELF WEAKLY SATURATED GRAPHS

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A graph G of order p with q edges is **self weakly saturated** *if* $wsat(p, G) = q - 1$.

SELF WEAKLY SATURATED GRAPHS

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A graph G of order p with q edges is **self weakly saturated** *if* $wsat(p, G) = q - 1$.

Definition

If a self weakly saturated graph G has a vertex v (called the **root***) and an ordering of the edges of $\overline{G}$ such that as each edge is added it is possible to choose a new copy of G in which v is always the same vertex, then the graph G is a* **rooted self weakly saturated graph***.*

SELF WEAKLY SATURATED GRAPHS

EXAMPLES

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(1) P_n is rooted self weakly saturated

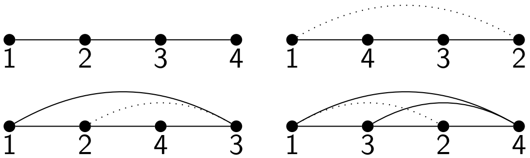

Figure: P_4 is a rooted self weakly saturated graph with vertex 1 as a root.

SELF WEAKLY SATURATED GRAPHS

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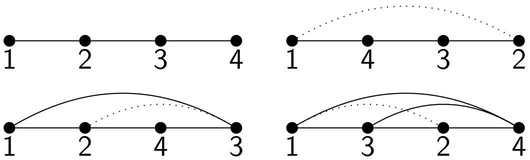

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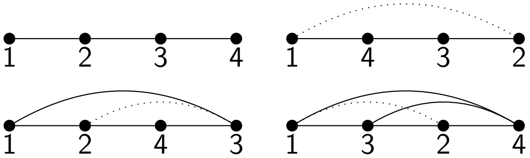

Figure: P_4 is a rooted self weakly saturated graph with vertex 1 as a root.

(2) $K_n - F_p$, when F_p is a forest with $p < n$ is self weakly saturated.

(3) $C_n \cup e$ where e is a 2-chord of C_n is self weakly saturated.

SELF WEAKLY SATURATED GRAPHS

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Let F be a graph of order p with q edges containing a cut-vertex v such that one of the components of $F - v$ along with v forms a rooted self saturated tree, T_m with root v. Then, for $n \geq 2p - m$,

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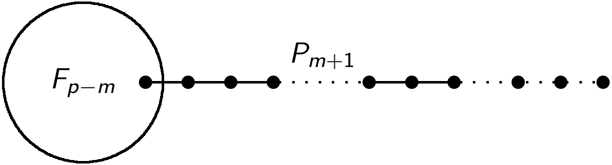

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For any given pair of positive integers p and q with $p - 1 \leq q \leq \binom{p}{2}$, there is a connected self weakly saturated graph with p vertices and q edges.

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Question

Given a graph G with p vertices and q edges, what conditions on the graphical parameters of G would imply that G is self weakly saturated.

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Theorem

If H is a vertex symmetric graph of order p with $\delta(H) = \delta$, then

$$wsat(2p, H - H) = \delta p.$$

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Question

Is there a universal lower bound for the weak saturation number in terms of the minimum degree of a graph? In particular is $ws(n, G) \geq (\delta n)/2 + c$ *for some constant c.*

THANKS

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QUESTIONS?