Some Problems in Measure Combinatorial Geometry Associated with Paul Erdős

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Measure combinatorial geometry

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Dear Falconer,

Can you prove that a set of Hausdorff dimension $2-\varepsilon$ contains an equilateral triangle for some $\varepsilon>0$. Also if r is large and you have a set in the circle of radius r of area $> r^{2-\varepsilon}$ must it then contain the vertices of an equilateral triangle of sides ≥ 1?

Large sets with no equilateral triangles

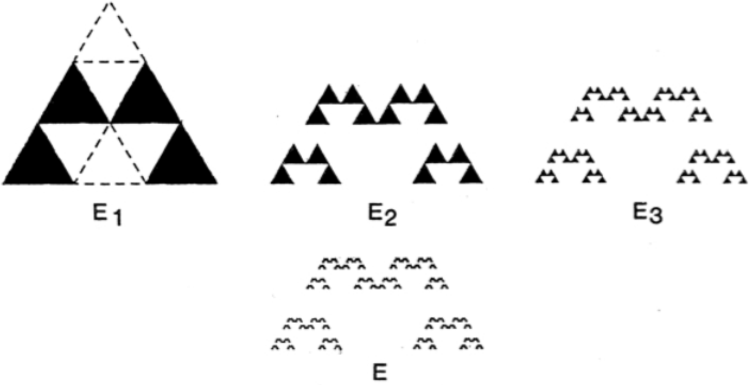
Question (Erdős, 1980s): How large can the Hausdorff dimension of a subset of $\mathbb{R}^2$ be if it does not contain the vertices of any equilateral triangle?

The Hausdorff dimension of $E \subset \mathbb{R}^N$ is given by

$$\dim_H E = \inf\Big\{s : \text{for all } \epsilon > 0 \text{ there is a countable cover } \{U_i\} \text{ of } E \text{ such that } \sum_i (\text{diam } U_i)^s < \epsilon\Big\}.$$

Caveat: All sets in this talk are assumed to be 'reasonable', i.e. Borel or analytic.

A self-similar set (taking open triangles at each step)



No equilateral triangle, $\dim_H E = \log 4/\log 3$.

Von Koch curve with vertex angle θ

No equilateral triangle for $\theta > \frac{1}{3}\pi$, $\dim_H E(\theta) \nearrow \log 4/\log 3$ as $\theta \nearrow \frac{1}{3}\pi$.

Is $\log 4/\log 3$ the largest dimension possible with for a set not containing the vertices of an equilateral triangle?

Proposition *There are plane sets of Hausdorff dimension 2 that do not contain the vertices of any equilateral triangle.*

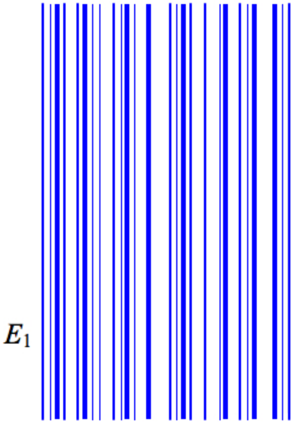
This depends on a construction due to Mattila (1984) which is very useful for counter-examples:

> There exist compact sets $E, F \subset [0,1]$ such that
>
> $$\dim_H E = \dim_H F = 1$$
>
> and $E \cap \tau(F)$ contains at most 1 point for all translations τ.

Equivalently, $D(E) \cap D(F) = \{0\}$ where $D(E)$ is the distance set of E,

$$D(E) = \{|x-y| : x, y \in E\}.$$

To construct a set of as required, let $E_1 = E \times \mathbb{R}$ and let F_1 be the set $F \times \mathbb{R}$ rotated through an angle of $\frac{1}{3}\pi$. Then $\dim_H E_1 = \dim_H F_1 = 2$, and by standard intersection properties, $\dim_H(E_1 \cap F_1) = 2$.



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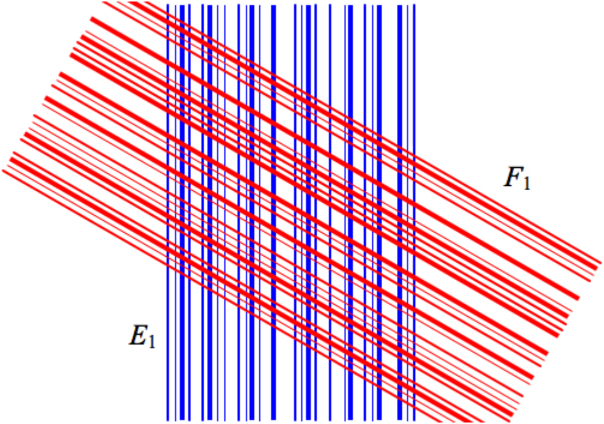
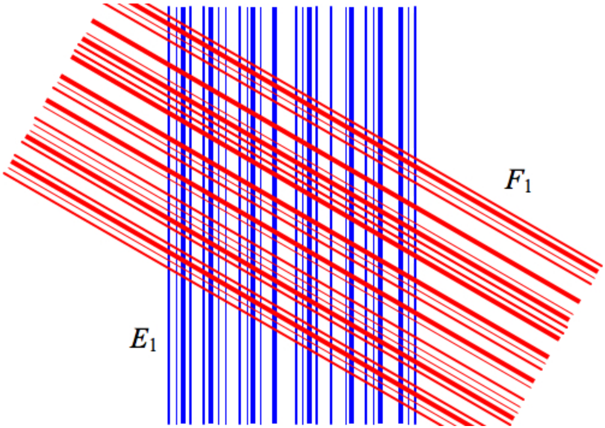


Then $E_1 \cap F_1$ is a suitable set:



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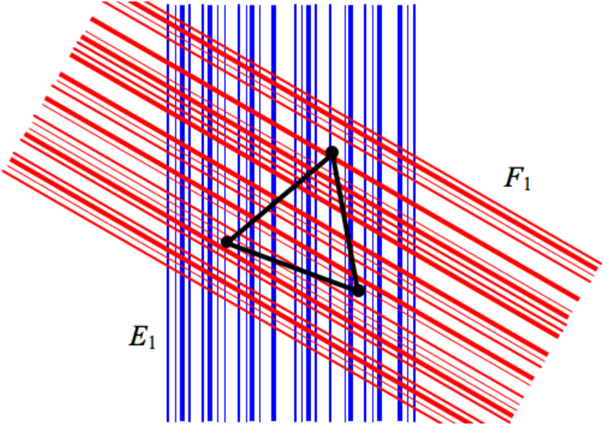

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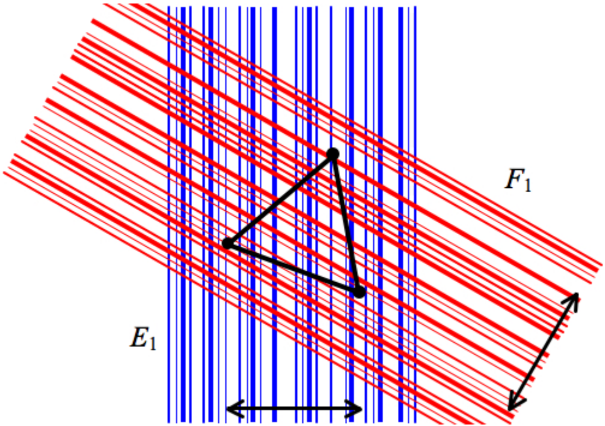


Given any equilateral triangle then the length of projection of one side onto the horizontal equals that of a second side onto the 60° line. If the vertices lie in $E_1 \cap F_1$ then this length lies in $D(E) \cap D(F) = \emptyset$.

• The same conclusion holds for any given triangle T , i.e. there exists a set E of dimension 2 such that no similar copy of T has all 3 vertices in E.

• How large can the Hausdorff dimension be of a self-similar subset of $\mathbb{R}^2$ containing no equilateral triangle?
(A self-similar set is one that is compact and satisfies $E = \cup_{i=1}^{m} f_i(E)$ where the f_i are contracting similarities)

• (Erdős) Is there a number $c > 0$ such that every measurable $E \subset \mathbb{R}^2$ with $\mathcal{L}^2(E) \geq c$ contains a triangle of area 1?

Distance sets

Erdős proposed his famous discrete distance problem in 1946 (or maybe earlier) and publicised it widely throughout his life:

> Let there be given n points in the plane
> $h(n)$ is the largest integer so that these points
> determine at least $h(n)$ distinct distances. An old
> conjecture of mine asserts
>
> (1) $$h(n) > c_1 \, n/\sqrt{\log n}$$
>
> I offered 500 dollars for a proof or disproof of (1).
> If true (1) is best possible. The lattice points in
> the plane show this.

Thus, with the distance set of $E \subset \mathbb{R}^n$ given by

$$D(E) = \{|x-y| : x, y \in E\},$$

the distance problem asked for a proof or disproof of

$$\#D(E) \geq c\frac{n}{\sqrt{\log n}} \qquad (1)$$

for all n-point sets E in the plane.

Erdős (1946) showed that $\#D(E) \geq cn^{1/2}$ and the exponent was gradually improved over the next 65 years.

The problem was (virtually) solved, with $\sqrt{\log n}$ replaced by $\log n$ in (1), a couple of years ago by Guth and Katz.

It seemed natural to look at a measure theoretic analogue and relate the *Hausdorff dimensions* of a set and its distance set.

Theorem (F, 1985)

Let $E \subseteq \mathbb{R}^2$ be a Borel set.
(*a*) $\dim_H E > 1\frac{1}{2} \Rightarrow \mathcal{L}(D(E)) > 0$.
(*b*) $\dim_H D(E) \geq \min\{1, \dim_H E - \frac{1}{2}\}$,

The proof of this (and most variations) uses Fourier transforms.

The largest known set with $\mathcal{L}^2(D(E)) = 0$ has $\dim_H E = 1$ obtained by a lattice based construction –
Do the lattice constructions give the least possible dimensions of $D(E)$?

Conjecture

$$\dim_H E > 1 \Rightarrow \mathcal{L}(D(E)) > 0,$$

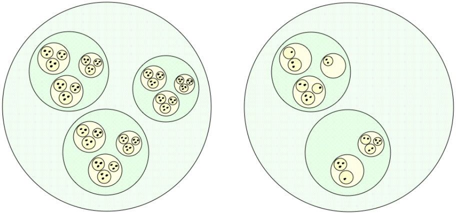
$$\dim_H D(E) \geq \min\{1, \dim_H E\}.$$

Best result known (Wolff, 1999)

$$\dim_H E > 1\tfrac{1}{3} \Rightarrow \mathcal{L}(D(E)) > 0.$$

Cases where $\dim_H D(E) \geq \min\{1, \dim_H E\}$:

• If E is a self-similar subset of $\mathbb{R}^2$, i.e. compact and satisfying $E = \cup_{i=1}^m f_i(E)$ where the f_i are contracting similarities. (See Hochman & Shemrkin 2011, Orponen, 2012, Farkas 2013 for different cases)



Self-similar set Percolation subset

• Let E_p be a percolation subset of a self-similar set, i.e. for a probability $p > 0$ each component in the hierarchical construction of a self-similar set is retained with probability p. Conditional on non-extinction, $\dim_H E_p$ takes a value d (which can be determined) almost surely. Then $\dim_H D(E) \geq \min\{1, d\}$ almost surely. (Rams & Simon 2013 for percolation based on squares, F & Jin 2013 for percolation on a general OSC self-similar set with a rotational component.)

What if instead of using the Euclidean metric on $\mathbb{R}^2$ we define distance with respect to some other norm? The Hausdorff dimension of $E \subset \mathbb{R}^2$ is independent of the norm, but $D(E)$ can change completely.

Curvature, or lack of it, is crucial here. Let $B = \{x : \|x\| \leq 1\}$ be the unit ball with respect to a given norm $\| \cdot \|$.

If B is strictly convex with nowhere vanishing curvature, then

$$\dim_H E > 1\tfrac{1}{2} \Rightarrow \mathcal{L}(D(E)) > 0. \quad \text{(Iosevich \& Łaba 2003)}$$

On the other hand, if B is a polygon, then there exists $E \subset \mathbb{R}^2$ with $\dim_H E = 2$ such that $\mathcal{L}(D(E)) = 0$.

This was shown for 'almost all' polygons by Iosevich & Łaba 2003. However, extending the approach indicated above for constructing sets of dimension 2 with no equilateral triangles, gives such examples for *any* norm for which the unit ball is a polygon (F, 2005).

Subrings of $\mathbb{R}$

Let $0 \le \alpha \le 1$. Volkmann and I proved in Crelle Journal (J für reine und angewandte Mathematik about 1965 or 6) that there is an ~~group~~ additive group of real numbers of Hausdorff dimension α. We could never decide if the same holds for rings, fields, or even algebraically closed fields.

Kind regards to you and your colleagues

P. Erdős

For each $0 < \alpha < 1$ does there exist a (Borel) subring E of $(\mathbb{R}, +, \times)$ with $\dim_H E = \alpha$?

This is related to the distance set problem: If E is a Borel ring, then

$$D(E \times E)^2 = \Big\{\mathrm{dist}\big((x_1, y_1), (x_2, y_2)\big)^2 : (x_i, y_i) \in E \times E\Big\}$$
$$= \Big\{(x_1 - x_2)^2 + (y_1 - y_2)^2 : x_i, y_i \in E\Big\} \subseteq E.$$

Then

$$\dim_H E \geq \dim_H D(E \times E)^2 = \dim_H D(E \times E)$$
$$\geq \min\{1, \dim_H(E \times E) - \tfrac{1}{2}\} \geq \min\{1, 2\dim_H E - \tfrac{1}{2}\}.$$

So either $\dim_H E = 1$ or $\dim_H E \leq \frac{1}{2}$,
i.e. there are no subrings of dimension α for $\frac{1}{2} < \alpha < 1$.

• If the general distance conjecture above were true then the same argument would give that there are no subrings of dimension α for $0 < \alpha < 1$.

• The non-existence of subrings of dimension α for $0 < \alpha < 1$ was eventually established by Edgar & Miller (2002) and Bourgain (2003) using more algebraic methods.

• It is crucial that we work with Borel or analytic sets here. Using CH one can construct a ring $E \subset \mathbb{R}$ with $\dim_H E = \alpha$ for all $0 < \alpha < 1$ (Davies 1984 ms).

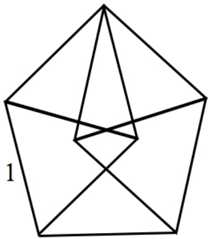
Chromatic number of the plane

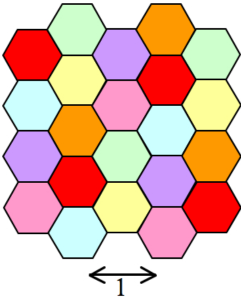
How many colours are needed to colour $\mathbb{R}^2$ so that no two points unit distance apart have the same colour?

The problem was formulated by Nelson in 1950 and promoted widely by Erdős and others.
Write χ for the least such number.

Current best bounds known: $4 \leq \chi \leq 7$.

Proof:





• By the theorem of de Bruijn & Erdős (1951), assuming AC it is enough to find the maximal chromatic number of a finite unit distance graph. But even to find a 5-chromatic embeddable graph may require millions of vertices and edges!

One approach is to consider particular classes of colourings, such as measurable colourings, that is when the set of points with each colour is plane Lebesgue measurable.

> I wanted to show since a long time that if one wants to determine the chromatic number of the plane (i.e. the chromatic number of the graph where two points of the plane are joined if their distance is one) it suffices to consider decomposition of the plane into measurable sets. This I can not do. If true it might be possible to prove that the chromatic number of the plane is > 4.

However, if C is a circle such that the angle subtended by a chord of unit length is not a rational multiple of π, then $\chi(C) = 2$ (using AC), but for a measurable colouring $\chi_M(C) = 3$ (using the ergodic theorem).

Theorem (F, 1981)

For measurable colourings of $\mathbb{R}^2$, *the chromatic number* $\chi_M \geq 5$.

Corollary

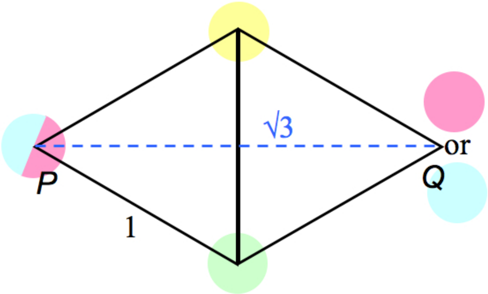
In the consistent system

$\{ZF + \textit{Countable AC} + \textit{All sets Lebesgue measurable}\}$

we have $\chi_M \geq 5$.

Sketch of proof:

Suppose there is a 4-colouring. If a double unit equilateral triangle has vertex P on the boundary of two colours, R and B, say, and the other vertices are not on the boundaries of colours, then Q must be coloured either R and B. Thus, the circle centre P and radius $\sqrt{3}$ is (essentially) coloured either R and B. But by an ergodic (or density) argument, this circle cannot be measurably 2-coloured.



By working with the density boundaries of the coloured sets (the set of points where the Lebesgue density is not 0 or 1), this can be made precise for measurable colourings.

Thank you!