THE MAHLER MEASURE OF THE RUDIN-SHAPIRO POLYNOMIALS

Tamás Erdélyi

Texas A&M University

1. Introduction

Let D be the open unit disk of the complex plane. Its boundary, the unit circle of the complex plane, is denoted by ∂D. Let

$$\mathcal{K}_n :=$$

$$\left\{ p_n : p_n(z) = \sum_{k=0}^{n} a_k z^k, \;\; a_k \in \mathbb{C}\,, \;\; |a_k| = 1 \right\} .$$

The class $\mathcal{K}_n$ is often called the collection of all (complex) unimodular polynomials of degree n. Let

$$\mathcal{L}_n := \left\{ p_n : p_n(z) = \sum_{k=0}^{n} a_k z^k, \;\; a_k \in \{-1, 1\} \right\} .$$

The class $\mathcal{L}_n$ is often called the collection of all Littlewood polynomials of degree n. By Parseval's formula,

$$\int_0^{2\pi} |P_n(e^{it})|^2 \, dt = 2\pi(n+1)$$

for all $P_n \in \mathcal{K}_n$.

Therefore

$$\min_{z \in \partial D} |P_n(z)| < \sqrt{n+1} < \max_{z \in \partial D} |P_n(z)|$$

for all $P_n \in \mathcal{K}_n$. An old problem (or rather an old theme) is the following.

Problem 1.1 (Littlewood's Flatness Problem). *Examine that how close a unimodular polynomial $P_n \in \mathcal{K}_n$ or $P_n \in \mathcal{L}_n$ can come to satisfying*

$$(1.1) \qquad |P_n(z)| = \sqrt{n+1}\,, \qquad z \in \partial D\,.$$

Obviously (1.1) is impossible if $n \geq 1$. So one must look for less than (1.1), but then there are various ways of seeking such an "approximate situation". One way is the following.

In his paper [Li1] Littlewood had suggested that, conceivably, there might exist a sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ (possibly even $P_n \in \mathcal{L}_n$) such that

$$(n+1)^{-1/2}|P_n(e^{it})|$$

converges to 1 uniformly in $t \in \mathbb{R}$. We shall call such sequences of unimodular polynomials "ultra-flat".

Definition 1.2. *Given a positive number ε, we say that a polynomial $P_n \in \mathcal{K}_n$ is ε-flat if*

$$(1-\varepsilon)\sqrt{n+1} \leq |P_n(z)| \leq (1+\varepsilon)\sqrt{n+1}\,, \qquad z \in \partial D\,.$$

Definition 1.3. *Given a sequence (ε_n) of positive numbers tending to 0, we say that a sequence (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$ is (ε_n)-ultraflat if*

$$(1-\varepsilon_n)\sqrt{n+1} \leq |P_n(z)| \leq (1+\varepsilon_n)\sqrt{n+1}\,, \qquad z \in \partial D\,.$$

The existence of an ultraflat sequence of unimodular polynomials seemed very unlikely, in view of a 1957 conjecture of P. Erdős (Problem 22 in [Er]) asserting that there is an absolute constant $\varepsilon > 0$ (independent of n) such that

$$\max_{z \in \partial D} |P_n(z)| \geq (1+\varepsilon)\sqrt{n+1}\,, \tag{1.2}$$

for all $P_n \in \mathcal{K}_n$ with $n \geq 1$. Yet, refining a method of Körner, Kahane (1985) proved that there exists a sequence (P_n) with $P_n \in \mathcal{K}_n$ which is (ε_n)-ultraflat, where

$$\varepsilon_n = O\left(n^{-1/17}\sqrt{\log n}\right)\,.$$

Thus the Erdős conjecture (1.2) was disproved for the classes $\mathcal{K}_n$. For the more restricted class $\mathcal{L}_n$ the analogous Erdős conjecture is unsettled to this date. It is a common belief that the analogous Erdős conjecture for $\mathcal{L}_n$ is true, and consequently there is no sequence of ultraflat unimodular polynomials $P_n \in \mathcal{L}_n$.

Based on certain properties satisfied by the ultraflat sequences of unimodular polynomials constructed by Kahane, in 1992 Saffari formulated a few conjectures on the behavior of all possible ultraflat sequences of unimodular polynomials.

Most of these conjectures, including Saffari's "phase problem" and Saffari's "near orthogonality conjecture", were proved in [E-01a], [E-01b], and [E-03].

A recent paper of Bombieri and Bourgain [BB] is devoted to the construction of ultraflat sequences of unimodular polynomials. In particular, one obtains a much improved estimate for the error term. A major part of this paper deals also with the long-standing problem of the effective construction of ultraflat sequences of unimodular polynomials.

A weaker version of Erdős's conjecture states that there is an absolute constant $\varepsilon > 0$ (independent of n) such that

$$\max_{z \in \partial D} |P_n(z)| \geq \sqrt{n+1} + \epsilon$$

for every $P_n \in \mathcal{L}_n$ with $n \geq 1$.

It is conjectured that there are sequences of flat Littlewood polynomials $P_n \in \mathcal{L}_n$ satisfying

$$c_1 \sqrt{n+1} \leq |P_n(z)| \leq c_2 \sqrt{n+1}$$

for all $z \in \mathbb{C}$ on the unit circle with absolute constants $c_1 > 0$ and $c_2 > 0$.

However, the lower bound part of this conjecture, by itself, seems hard, and no sequence is known that satisfies just the lower bound. A sequence of Littlewood polynomials satisfying just the upper bound is given by the Rudin-Shapiro polynomials. They appear in Harold Shapiro's 1951 thesis at MIT and are sometimes called just Shapiro polynomials.

They also arise independently in Golay's paper [Go-51]. They are remarkably simple to construct and are a rich source of counterexamples to possible conjectures. The Rudin-Shapiro polynomials are defined recursively as follows:

$$P_0(z) := 1\,, \qquad Q_0(z) := 1\,,$$

and

$$P_{n+1}(z) := P_n(z) + z^{2^n} Q_n(z)\,,$$
$$Q_{n+1}(z) := P_n(z) - z^{2^n} Q_n(z)$$

for $n = 0, 1, 2, \ldots$. Note that both P_n and Q_n are polynomials of degree $N-1$ with $N := 2^n$ having each of their coefficients in $\{-1, 1\}$. It is well known and easy to check by using the parallelogram law that

$$|P_{n+1}(z)|^2 + |Q_{n+1}(z)|^2 = 2(|P_n(z)|^2 + |Q_n(z)|^2)$$

for every $z \in \partial D$. Hence

$$|P_n(z)|^2 + |Q_n(z)|^2 = 2^{n+1} = 2N\,, \quad z \in \partial D\,,$$

and

$$|P_n(z)| \le \sqrt{2N}\,, \quad z \in \partial D\,.$$

Let $\alpha < \beta$ be real numbers. The Mahler measure $M_0(P, [\alpha, \beta])$ is defined for bounded measurable functions P defined on $[\alpha, \beta]$ as

$$M_0(P, [\alpha, \beta]) := \exp\left(\frac{1}{\beta - \alpha}\int_\alpha^\beta \log|P(e^{it})|\,dt\right).$$

It is well known that

$$M_0(P, [\alpha, \beta]) = \lim_{q \to 0+} M_q(P, [\alpha, \beta]),$$

where, for $q > 0$,

$$M_q(P, [\alpha, \beta]) := \left(\frac{1}{\beta - \alpha}\int_\alpha^\beta \left|P(e^{it})\right|^q\,dt\right)^{1/q}.$$

It is a simple consequence of the Jensen formula that

$$M_0(P) := M_0(P, [0, 2\pi]) = |c|\prod_{k=1}^{n}\max\{1, |z_k|\}$$

for every polynomial of the form

$$P(z) = c\prod_{k=1}^{n}(z - z_k), \qquad c, z_k \in \mathbb{C}.$$

P. Borwein's book [B-02] presents a few more basic results on the Rudin-Shapiro polynomials. Various other properties of them are discussed in [BLM-76]. Obviously

$$M_2(P_n, [0, 2\pi]) = \sqrt{N}\,, \quad N := 2^n,$$

by the Parseval formula. In 1969 Littlewood [Li-68] evaluated $M_4(P_n, [0, 2\pi])$ and found that

$$M_4(P_n, [0, 2\pi]) \sim (4N^2/3)^{1/4}\,, \quad N := 2^n.$$

Rudin-Shapiro like polynomials in L_4 on the unit circle of the complex pane are studied in [BM-00]. In 1980 Saffari [Sa-01] conjectured that with $N := 2^n$ we have

$$M_q(P_n, [0, 2\pi]) \sim \frac{\sqrt{2N}}{(q/2+1)^{1/q}}\,, \qquad q > 0\,.$$

P. Borwein and Lockhart [BL-01] investigated the asymptotic behavior of the mean value of normalized L_p norms of Littlewood polynomials for arbitrary $p > 0$. Using the Lindeberg Central Limit Theorem and dominated convergence, they proved that

$$\lim_{n \to \infty} \frac{1}{2^{n+1}} \sum_{f \in \mathcal{L}_n} \frac{(M_p(f, [0, 2\pi]))^p}{n^{p/2}} = \Gamma\left(1 + \frac{p}{2}\right) .$$

The paper [CM-11] establishes beautiful results on the average value of the Mahler measure and L_p norms of unimodular polynomials in $\mathcal{K}_n$.

2 Mahler measure and moments of the Rudin-Shapiro polynomials

Our first theorem states that the Mahler measure and the maximum norm of the Rudin-Shapiro polynomials on the unit circle of the complex plane have the same size.

Theorem 2.1. *Let P_n and Q_n be the n-th Rudin-Shapiro polynomials defined in Section 1. There is an absolute constant $c_1 > 0$ such that*

$$M_0(P_n, [0, 2\pi]) = M_0(Q_n, [0, 2\pi]) \geq c_1 \sqrt{N}\,,$$

where

$$N := 2^n = \deg(P_n) + 1 = \deg(Q_n) + 1\,.$$

To formulate our next theorem we define

$$\widetilde{P}_n := 2^{-(n+1)/2} P_n \quad \text{and} \quad \widetilde{Q}_n := 2^{-(n+1)/2} Q_n \,.$$

By using the above normalization, we have

$$|\widetilde{P}_n(z)|^2 + |\widetilde{Q}_n(z)|^2 = 1 \,, \qquad z \in \partial D \,.$$

For $q > 0$ let

$$\begin{aligned} I_q(\widetilde{P}_n) :=& (M_q(\widetilde{P}_n, [0, 2\pi]))^q \\ :=& \frac{1}{2\pi} \int_0^{2\pi} |\widetilde{P}_n(e^{i\tau})|^q \, d\tau \,. \end{aligned}$$

The following result is a simple consequence of Theorem 2.1.

Theorem 2.2. *There exists a constant $L < \infty$ independent of n such that*

$$\sum_{k=1}^{\infty} \frac{I_k(\widetilde{P}_n)}{k} < L \,, \qquad n = 0, 1, \ldots \,.$$

Theorem 2.3. *There exists an absolute constant $c_2 > 0$ such that*

$$M_0(P_n, [\alpha, \beta]) \geq c_2 \sqrt{N}$$

for all $n \in \mathbb{N}$ with $N := 2^n$, and for all $\alpha, \beta \in \mathbb{R}$ such that

$$\frac{12\pi}{N} \leq \frac{(\log N)^{3/2}}{N^{1/2}} \leq \beta - \alpha \leq 2\pi\,.$$

It looks plausible that Theorem 2.3 holds when $12\pi/N \leq \beta - \alpha \leq 2\pi$, but we do not seem to be able to handle the case

$$12\pi/N \leq \beta - \alpha \leq (\log N)^{3/2} N^{-1/2}$$

at the moment.

3. Lemmas

A key to the proof of Theorem 2.1 is the following observation which is a straightforward consequence of the definition of the Rudin Shapiro polynomials P_n and Q_n.

Lemma 3.1. *Let* $n \geq 2$, $N := 2^n$, *and*

$$z_j := e^{it_j}, \quad t_j := \frac{2\pi j}{N}, \quad j = 0, 1, \ldots, N-1.$$

We have

$$P_n(z_j) = 2P_{n-2}(z_j)\,, \qquad j = 2u,$$
$$P_n(z_j) = (-1)^{\frac{j-1}{2}} 2i\, Q_{n-2}(z_j)\,, \qquad j = 2u+1\,,$$

for $u = 0, 1, \ldots, N/2 - 1$.

Combining this with a "Riesz Lemma type" polynomial inequality we obtain the following.

Lemma 3.2. *Let P_n and Q_n be the n-th Rudin-Shapiro polynomials. Let $N := 2^n$, $\gamma := \sin^2(\pi/8)$, and*

$$z_j := e^{it_j}, \quad t_j := \frac{2\pi j}{N}, \quad j = 0, 1, \ldots, N-1.$$

We have

$$\max\{|P_n(z_j)|^2, |P_n(z_{j+1})|^2\} \geq 2\gamma N$$

for all $j = 2u$, $u = 0, 1, \ldots, N/2 - 1$.

Another key to the proof of Theorem 2.1 is a sieve-type lower bound for the Mahler measure of polynomials proved in [EL-07]. Let $\mathcal{P}_N$ be the set of all polynomials of degree at most N with real coefficients.

Lemma 3.3. *Assume that* $N, m \geq 1$*,*

$$0 < \tau_1 \leq \tau_2 \leq \cdots \leq \tau_m \leq 2\pi\,,$$

$$\tau_0 := \tau_m - 2\pi\,, \quad \tau_{m+1} := \tau_1 + 2\pi\,.$$

Let

$$\delta := \max\{\tau_1 - \tau_0, \tau_2 - \tau_1, \ldots, \tau_m - \tau_{m-1}\}\,.$$

For every $A > 0$ *there is a* $B > 0$ *depending only on* A *such that*

$$\sum_{j=1}^{m} \frac{\tau_{j+1} - \tau_{j-1}}{2} \log |P(e^{i\tau_j})|$$

$$\leq \int_0^{2\pi} \log |P(e^{i\tau})|\, d\tau + B$$

for all $P \in \mathcal{P}_N$ *and* $\delta \leq AN^{-1}$*.*

4. The Mahler measure of the Fekete polynomials

Finding polynomials with suitably restricted coefficients and maximal Mahler measure has interested many authors.

Beller and Newman [BN-73] constructed unimodular polynomials of degree n whose Mahler measure is at least $\sqrt{n} - c/\log n$. For a prime number p the p-th Fekete polynomial is defined as

$$f_p(z) := \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) z^k \,,$$

where

$$\left(\frac{k}{p}\right) = \begin{cases} 1, & \text{if } \; x^2 \equiv k \pmod{p} \; \text{ for an } \; x \neq 0\,, \\ 0, & \text{if } \; p \; \text{divides} \; k\,, \\ -1, & \text{otherwise} \end{cases}$$

is the usual Legendre symbol.Note that $g_p(z) := f_p(z)/z$ is a Littlewood polynomial, and has the same Mahler measure as f_p.

In [Mo-80] Montgomery proved the following fundamental result.

Theorem 4.1. *There are absolute constants* $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$c_1\sqrt{p}\log\log p \le \max_{z\in\partial D}|f_p(z)| \le c_2\sqrt{p}\log p\,.$$

In [EL-07] we proved the following result.

Theorem 4.2. *For every* $\varepsilon > 0$ *there is a constant* c_ε *such that*

$$M_0(f_p,[0,2\pi]) \ge \left(\frac{1}{2}-\varepsilon\right)\sqrt{p}$$

for all primes $p \ge c_\varepsilon$.

One of the key lemmas in the proof of the above theorem formulates a remarkable property of the Fekete polynomials. A simple proof of it is given in [B-02, pp. 37-38].

Lemma 4.3 (Gauss). *We have*

$$f_p(z_p^j) = \varepsilon_p \left(\frac{j}{p}\right) p^{1/2}\,, \qquad j = 1, 2, \dots, p-1\,,$$

and $f_p(1) = 0$*, where*

$$z_p := \exp\left(\frac{2\pi i}{p}\right)$$

is the first p-th root of unity, and $\varepsilon_p \in \{\pm 1, \pm i\}$*.*

The choice of ε_p is more subtle. This is also a result of Gauss, see [Hua-82].

Lemma 4.4 (Gauss). *In Lemma 4.3 we have*

$$\varepsilon_p = \begin{cases} 1, & \text{if} \quad p \equiv 1 \pmod 4 \\ i, & \text{if} \quad p \equiv 3 \pmod 4\,. \end{cases}$$

In [E-11] Theorem 4.2 is extended to subarcs of the unit circle.

Theorem 4.5. *There exists an absolute constant* $c_1 > 0$ *such that*

$$M_0(f_p, [\alpha, \beta]) \geq c_1 p^{1/2}$$

for all prime numbers p *and for all* $\alpha, \beta \in \mathbb{R}$ *such that* $(\log p)^{3/2} p^{-1/2} \leq \beta - \alpha \leq 2\pi$.

In [E-12] we gave an upper bound for the average value of $|f_p(z)|^q$ over any subarc I of the unit circle, valid for all sufficiently large primes p and all exponents $q > 0$.

Theorem 4.6. *There exists a constant* $c_2(q, \varepsilon)$ *depending only on* $q > 0$ *and* $\varepsilon > 0$ *such that*

$$M_q(f_p, [\alpha, \beta]) \leq c_2(q, \varepsilon) p^{1/2},$$

for all prime numbers p *and for all* $\alpha, \beta \in \mathbb{R}$ *such that* $\beta - \alpha \geq 2p^{-1/2+\varepsilon}$.

We remark that a combination of Theorems 4.5 and 4.6 shows that there is an absolute constant $c_1 > 0$ and a constant $c_2(q, \varepsilon) > 0$ depending only on $q > 0$ and $\varepsilon > 0$ such that

$$c_1 p^{1/2} \le M_q(f_p, [\alpha, \beta]) \le c_2(q, \varepsilon) p^{1/2}$$

for all prime numbers p and for all $\alpha, \beta \in \mathbb{R}$ such that $\beta - \alpha \ge 2p^{-1/2+\varepsilon} \ge (\log p)^{3/2} p^{-1/2}$.

Conrey, Granville, Poonen, and Soundararajan (2000) showed that f_p has asymptotically κp zeros on the unit circle, where

$$0.500668 < \kappa < 0.500813\,.$$