Gelfond Problems on the Sum-of-digits Function and Subsequences of Automatic Sequences

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Sehr geehrter Herr Dr Drmota,

Vielen Dank für Ihre lange und interessante Arbeit. Vielleicht kennen Sie folgendes alte Problem (Ich stellte die Frage in 1931!) Es sei

$$1 \le a_1 < a_2 < \dots < a_k \le n, \quad k \text{ maximal}$$

Die Summen $\sum_{i=1}^{k} \varepsilon_i a_i$ seien alle verschieden, $\varepsilon_i = 0$ oder 1 ($a_i = 2^i$ ist eine Lösung). Ist es aber wahr dass

$$k < \frac{\log n}{\log 2} + c$$

ist? Vielleicht ist $c = 3$.

Eine andere Frage: $f(n)$ sei multiplicativ, $f(n) = \pm 1$,

$$\sum_{f(p)=-1} \frac{1}{p} = \infty$$

(also ist $f(p) = -1$ ziemlich oft (p Primzahl)).

Ist es wahr dass $\sum_{n=1}^{x} f(n)$ unendlich oft 0 ist ($\overline{\lim} = +\infty$, $\underline{\lim} = -\infty$) wenn falsch genügt vielleicht

$$\sum_{\substack{f(p)=-1 \\ p<x}} 1 > c\,x/\log x \; ?$$

Es sei $1 < a_1 < a_2 < \dots < a_k \le n$, $[a_i, a_j] > n$, (also keine Zahl $m \le n$ ist ein vielfaches von 2 der a_i). Ist es wahr dass für $n > n_0$ $\sum \frac{1}{a_i} < 1$ ist? Wieviel Zahlen $\le n$ muss es geben die durch keine der a_i teilbar ist? $c\sqrt{n}$ ist trivial vielleicht gilt εn.

(Schinzel-Szekeres Sur un problème de Paul Erdős, Acta Szeged etwa 1960

Viele Grüsse
Ihr Paul Erdős

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Summary

- ★ Binary Expansion of Primes
- ★ Thue-Morse sequence
- ★ Gelfond problems
- ★ Generalized Thue-Morse sequence and uniform distribution
- ★ Automatic sequences
- ★ Generalized Thue-Morse sequence of squares
- ★ The sum-of-digits function of polynomial values
- ★ The sum-of-digits function of primes

★ Binary Representation of Primes

2 extremal cases

$p = 2^k + 1$ Fermat prime ($k = 2^m$)

$p = 2^k + 2^{k-1} + \cdots + 2 + 1$ Mersenne prime ($k + 1 \in \mathbb{P}$)

★ Binary Representation of Primes

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Question on the number of digits of primes (related to Bourgain and Green)

Given k, does there exist a prime p and $0 = j_1 < j_2 < \cdots < j_k$ with

$$p = 2^{j_1} + 2^{j_2} + \cdots + 2^{j_k} \text{ ??}$$

(This is equivalent to $s_2(p) = k$.)

★ Binary Representation of Primes

Answer for large k

Theorem (D., Mauduit, Rivat (2011))

$s_2(n)$ ... *number of powers of 2 in the binary expansion of* n

$$\#\{primes\ p < 2^{2k} : s_2(p) = k\} \sim \frac{2^{2k}}{\sqrt{2\pi}\log 2\, k^{\frac{3}{2}}}$$

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Unfortunately this does not give a proper answer for $k = 2$ or $k = \lfloor \log_2 p \rfloor$.

★ Thue-Morse sequence

Thue-Morse sequence $(t_n)_{n \geqslant 0}$:

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Thue-Morse sequence $(t_n)_{n \geqslant 0}$:

0

★ Thue-Morse sequence

Thue-Morse sequence $(t_n)_{n\geqslant 0}$:

01

★ Thue-Morse sequence

Thue-Morse sequence $(t_n)_{n\geqslant 0}$:

0110

★ Thue-Morse sequence

Thue-Morse sequence $(t_n)_{n \geqslant 0}$:

01101001

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$$0110100110010110$$

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$$t_0 = 0, \qquad t_{2^n+k} = 1 - t_k \quad (0 \leqslant k < 2^n)$$

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$$t_0 = 0, \qquad t_{2^n+k} = 1 - t_k \quad (0 \leqslant k < 2^n)$$

$$\boxed{t_n = s_2(n) \bmod 2}$$

$$n = \sum_{i=0}^{\ell-1} \varepsilon_i(n) q^i \qquad \varepsilon_i(n) \in \{0, 1, \ldots, q-1\}, \qquad s_q(n) = \sum_{i=0}^{\ell-1} \varepsilon_i(n)$$

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$$\#\{0\leqslant n<N : t_n=0\}\sim\frac{N}{2}$$

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$$\# \{0 \leqslant n < N : t_n = 0\} \sim \frac{N}{2}$$

$$\# \{0 \leqslant n < N : t_{3n} = 0\} \sim \frac{N}{2}$$

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$$10\ 0\ 1\quad 1\ 1\quad 0\ 1\quad 0\quad\quad 0\ 1\quad\quad 1\quad 1\ 0\ \cdots$$

Mauduit and Rivat (2010):

$$\#\{0 \leqslant p < N : t_p = 0\} \sim \frac{\pi(N)}{2}$$

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Mauduit and Rivat (2009):

$$\#\{0 \leqslant n < N : t_{n^2} = 0\} \sim \frac{N}{2}$$

★ Gelfond Problems

Gelfond 1967/1968

a, m ... positive integers, b, ℓ ... non-neg. integer, $(m, q-1) = 1$.

$$\Longrightarrow \quad \boxed{\#\{n < N : \ s_q(an+b) \equiv \ell \bmod m\} = \frac{N}{m} + O(N^\lambda)}$$

with $0 < \lambda < 1$.

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In particular:

$$\begin{aligned} \#\{n < N : t_{an+b} = 0\} &= \#\{n < N : s_2(an+b) \equiv 0 \bmod 2\} \\ &= \frac{N}{2} + O(N^\lambda) \end{aligned}$$

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1. $q_1, q_2, \ldots, q_d \geqslant 2$, $(q_i, q_j) = 1$ for $i \ne j$, $(m_j, q_j - 1) = 1$:

$$\#\{n < N : s_{q_j}(n) \equiv \ell_j \bmod m_j,\ 1 \leqslant j \leqslant d\} = \frac{N}{m_1 \cdots m_d} + O(N^{1-\eta})$$

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Drmota, Mauduit, Rivat 2011 for **large bases** $q \geqslant q_0(\deg(P))$

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★ Generalized Thue-Morse sequences

- H ... compact (Hausdorff) group
- $q \geqslant 2$ and $g_0, g_1, \ldots, g_{q-1} \in H$ with $g_0 = e$ (identity element)
- $G \leqslant H$ ... closure of the subgroup generated by $g_0, g_1, \ldots, g_{q-1}$

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Generalized Thue-Morse sequence:

$$\boxed{T(n) := g_{\varepsilon_0(n)} g_{\varepsilon_1(n)} \cdots g_{\varepsilon_{\ell-1}(n)}}$$

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$$n = \sum_{i=0}^{\ell-1} \varepsilon_i(n) q^i$$

q-multiplicative function:

$$\boxed{T(j + qn) = g_j T(n) = T(j)T(n)} \qquad 0 \leqslant j < q$$

★ Generalized Thue-Morse sequences

Examples

- $H = \langle \mathbb{Z}/2\mathbb{Z}, +\rangle$, $q = 2$, $g_0 = 0$, $g_1 = 1$:

$$\boxed{T(n) = s_2(n) \bmod 2 = t_n}.$$

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- $H = \langle \mathbb{R}/\mathbb{Z}, + \rangle$, $g_j = \alpha j$ $(0 \leqslant j < q)$:

$$\boxed{T(n) = \alpha\, s_q(n) \bmod 1}.$$

★ Generalized Thue-Morse sequences

Theorem

Let μ denote the Haar measure of G. Then $(T(n))_{n\geqslant 0}$ is μ-uniformly distributed in G, that is,

$$\frac{1}{N}\sum_{n=0}^{N-1}\delta_{T(n)} \to \mu.$$

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$$\frac{1}{N}\sum_{n=0}^{N-1} f(T(n)) \to \int_G f\,\mathrm{d}\mu$$

(for all continuous functions $f : G \to \mathbb{R}$.)

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(for all continuous functions $f : G \to \mathbb{R}$.)

Remark. Equivalently, a sequence (x_n) in G is μ-uniformly distributed if

$$\frac{1}{N} \left|\{n < N : x_n \in B\}\right| \to \mu(B)$$

holds for all μ-measurable sets $B \subseteq G$ with $\mu(\partial B) = 0$.

★ Generalized Thue-Morse sequences and Gelfond's 1st problem

Theorem

Let $q_1, \ldots, q_d \geqslant 2$, $(q_i, q_j) = 1$ *for* $i \neq j$ *and* $T_j(n)$, $1 \leqslant j \leqslant d$, *generalized Thue-Morse sequences with respect to* q_j *on groups* G_j *(with Haar measure* μ_j*).*
Then $(T_1(n), \ldots, T_d(n))$ *is* μ*-uniformly distributed in* $G = G_1 \times \cdots \times G_d$, *where* μ *denotes the product measure on* G*:*

$$\frac{1}{N} \sum_{n=0}^{N-1} \delta_{(T_1(n),\ldots,T_d(n))} \to \mu.$$

Theorem (D. and Morgenbesser, 2012)

Let $a \geqslant 1$ *and* $b \geqslant 0$ *be integers and set* $m' = \gcd(a, m)$ *(where* $m = m(q, g_0, \ldots, g_{q-1})$ *can be defined in a proper way). Set*

$$\mathrm{d}\nu' = m' \cdot \mathbf{1}_{T(b)U'}\,\mathrm{d}\mu,$$

where

- $\mu \ldots$ *Haar measure on G,*
- $U' = \mathrm{cl}(\{T(m'n) : n \geqslant 0\}) \ldots$ *normal subgroup of G of index* m'.

Then $(T(an+b))_{n\geqslant 0}$ *is* ν'*-uniformly distributed in G, that is,*

$$\boxed{\frac{1}{N}\sum_{n=0}^{N-1} \delta_{T(an+b)} \to \nu'}.$$

Theorem (D. and Morgenbesser, 2012)

There exists a positive integer $m = m(q, g_0, \ldots, g_{q-1})$ such that the following holds: Set

$$\mathrm{d}\nu = \sum_{v=0}^{m} \mathbf{1}_{g_v U} \cdot Q(v,m)\, \mathrm{d}\mu,$$

where

- $\mu \ldots$ *Haar measure on G,*
- $U = \mathrm{cl}(\{T(mn) : n \geqslant 0\}) \ldots$ *normal subgroup of G of index m,*
- $Q(v,m) = \#\{0 \leqslant n < m : n^2 \equiv v \bmod m\}$.

Then $(T(n^2))_{n\geqslant 0}$ is ν-uniformly distributed in G, that is,

$$\boxed{\frac{1}{N}\sum_{n=0}^{N-1} \delta_{T(n^2)} \to \nu}.$$

A **unitary group representation** is a continuous homomorphism $D : G \to U_n$ for some $n \geqslant 1$.

$U_n \ldots$ group of unitary $n \times n$ matrices over $\mathbb{C}$

D is irreducible if there is no proper subspace W of $\mathbb{C}^n$ with $D(x)W \subseteq W$ for all $x \in G$

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Lemma

Let G be a compact group and ν a regular normed Borel measure on G. Then a sequence $(x_n)_{n\geqslant 0}$ is ν-uniformly distributed in G iff

$$\frac{1}{N}\sum_{n=0}^{N-1} D(x_n) \to \int_G D\, \mathrm{d}\nu$$

holds for all irreducible unitary representations D of G.

Remarks:

- The integer $m = m(q, g_0, \ldots, g_{q-1})$ is the largest integer such that $m \mid q-1$ and such that there exists a representation D of G with

$$D(g_u) = e^{-2\pi i \frac{u}{m}} \qquad \text{for all } u \in \{0, 1, \ldots, q-1\}.$$

- $(T(n^2))_{n \geqslant 0}$ is uniformly distributed in G (i.e., $\nu = \mu$) iff $m \leqslant 2$.
- $(T(an+b))_{n \geqslant 0}$ is uniformly distributed in G (i.e., $\nu' = \mu$) iff $m' = \gcd(a, m) = 1$.

- If G is connected, then $T(n^2)$ and $T(an+b)$ are uniformly distributed in G.

★ Properties of the Fourier term

$$T_\lambda(n) := g_{\varepsilon_0(n)} g_{\varepsilon_1(n)} \cdots g_{\varepsilon_{\lambda-1}(n)} \quad \text{(periodic with period } q^\lambda\text{)}$$

$$F_\lambda(h) := \frac{1}{q^\lambda} \sum_{0 \leqslant u < q^\lambda} e^{-2\pi i \frac{hu}{q^\lambda}} D(T_\lambda(u))$$

Lemma

Set

$$\Psi_D(t) = \sum_{0 \leqslant u < q} e(tu) D(g_u),$$

then

$$F_\lambda(h) = \frac{1}{q^\lambda} \Psi_D\left(-\frac{h}{q^\lambda}\right) \Psi_D\left(-\frac{h}{q^{\lambda-1}}\right) \cdots \Psi_D\left(-\frac{h}{q}\right).$$

★ Properties of the Fourier term

Lemma

Suppose that $D \notin \{D_0, \ldots, D_{m-1}\}$ is an irreducible and unitary representation of G. Then there exists a constant $c > 0$ such that

$$\boxed{\max_{h\in\mathbb{Z}} \|F_\lambda(h)\|_2 \ll q^{-c\lambda}}.$$

★ Exercise on linear subsequences

$$\sum_{n<N} D(T(an+b))$$

$$= \sum_{0\leqslant u<q^\nu} \sum_{n<N} D(T(u)) \cdot \frac{1}{q^\lambda} \sum_{0\leqslant h<q^\lambda} \mathrm{e}\left(\frac{h(an+b-u)}{q^\lambda}\right)$$

$$= \sum_{0\leqslant h<q^\lambda} F_\lambda(h) \sum_{n<N} \mathrm{e}\left(\frac{h(an+b)}{q^\lambda}\right).$$

★ Exercise on linear subsequences

$$\sum_{n<N} D(T(an+b))$$
$$= \sum_{0\leqslant u<q^\nu} \sum_{n<N} D(T(u)) \cdot \frac{1}{q^\lambda} \sum_{0\leqslant h<q^\lambda} \mathrm{e}\left(\frac{h(an+b-u)}{q^\lambda}\right)$$
$$= \sum_{0\leqslant h<q^\lambda} F_\lambda(h) \sum_{n<N} \mathrm{e}\left(\frac{h(an+b)}{q^\lambda}\right).$$

$$\left\|\sum_{n<N} D(T(an+b))\right\|_2 \ll \sum_{0\leqslant h<q^\lambda} \|F_\lambda(h)\|_2 \cdot \min\left(N, \frac{1}{\left|\sin\frac{\pi ha}{q^\lambda}\right|}\right).$$

★ Sketch of the proof for squares

$$\boxed{\frac{1}{N} \sum_{0 \leqslant n < N} D(T(n^2))}$$

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- The representation $D_0, \ldots, D_{m-1}$ are special but easy:

$$D_k(g_u) = e^{-2\pi i \frac{k}{m} u} \qquad \text{for all } 0 \leqslant u < q \text{ and } 0 \leqslant k < m$$

$$D_k(T(n^2)) = e^{-2\pi i \frac{k}{m} n^2} \qquad \text{Gauss sums}$$

- For all other irreducible unitary representations ...

Van der Corput type inequality:

$$\left\| \sum_{0 \leqslant n < N} Z(n) \right\|_{\mathbb{F}} \leqslant \left(\frac{dN}{R} \sum_{|r| < R} \left(1 - \frac{|r|}{R}\right) \left\| \sum_{\substack{0 \leqslant n \leqslant B \\ 0 \leqslant n+r \leqslant B}} Z(n+r) Z(n)^H \right\|_{\mathbb{F}} \right)^{1/2} + \frac{f}{2} R$$

Van der Corput type inequality:

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Fourier terms:

$$F_{\lambda}(h) = \frac{1}{q^\lambda} \sum_{0 \leqslant u < q^\lambda} e^{-2\pi i \frac{hu}{q^\lambda}} D(T_{\lambda}(u))$$

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$$\cdot \sum_{\substack{0\leqslant h_1,h_2,h_3,h_4<q^{\lambda}\\ (h_1+h_2+h_3+h_4,q^{\lambda})=d \\ d\mid 2r(h_1+h_2)+2sq^{\mu}(h_2+h_3)+\ell}} \|F_{\mu,\lambda}(h_1)\|_{\mathbb{F}}\, \|F_{\mu,\lambda}(h_2)\|_{\mathbb{F}}\, \|F_{\mu,\lambda}(h_3)\|_{\mathbb{F}}\, \|F_{\mu,\lambda}(h_4)\|_{\mathbb{F}}$$

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This term can be estimated by applying upper bounds on the Fourier terms (in a quite subtle way; an analogue of this expression appears in Mauduit and Rivat's work).

★ Applications

★ The sum-of-digits function:

Theorem (Mauduit and Rivat, 2009)

Let $q, m \geqslant 2$ *and set* $r = \gcd(q-1, m)$. *Then we have*

$$\boxed{\lim_{N\to\infty} \frac{1}{N} \# \{n < N : s_q(n^2) \equiv a \bmod m\} = \frac{1}{r} Q(a,r)},$$

where $Q(a,r) = \#\{0 \leqslant n < r : n^2 \equiv a \bmod r\}$.
Furthermore, $(\alpha s_q(n^2))_{n\in\mathbb{N}}$ *is uniformly distributed modulo* 1 *iff* α *is irrational.*

★ Applications

★ Invertible automatic sequences:

Theorem (D. and Morgenbesser, 2012)

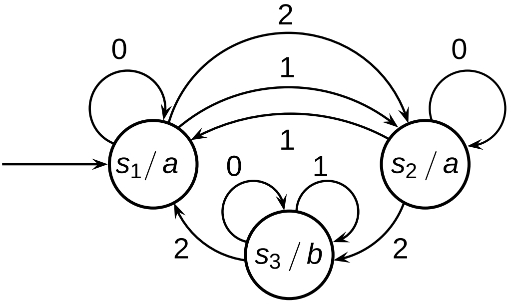
Let $q \geqslant 2$ and $(u_n)_{n\geqslant 0}$ be an invertible q-automatic sequence. Then the **frequency** *of each letter of the subsequence $(u_{n^2})_{n\geqslant 0}$ exists.*

Remark. In invertible automatic sequences the frequencies of the stated in the automaton is equidistributed. For the subsequenc of squares the limiting distribution might be different.

★ Automatic sequences

Definition

A sequence $(u_n)_{n \geqslant 0}$ is called a *q-automatic sequence*, if u_n is the output of an automaton when the input is the q-ary expansion of n.



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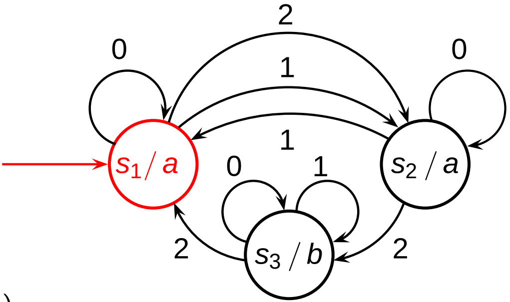

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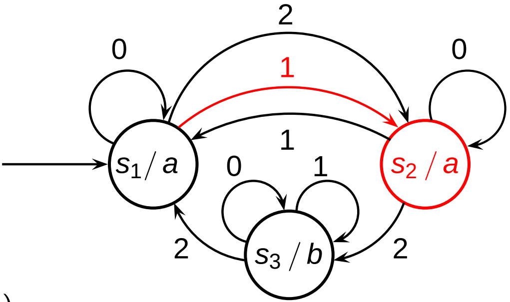

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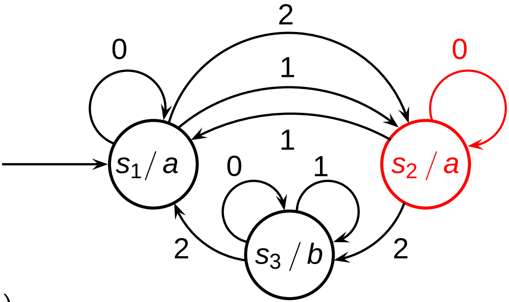

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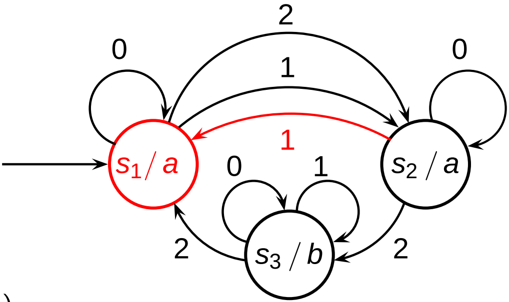

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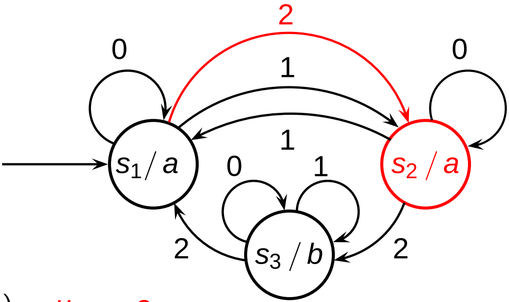

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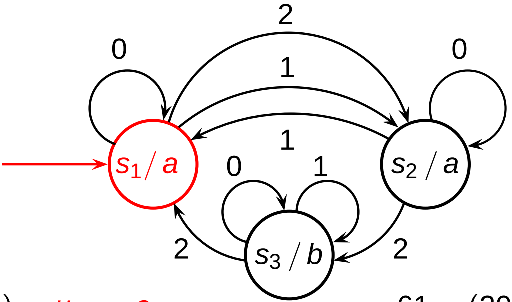

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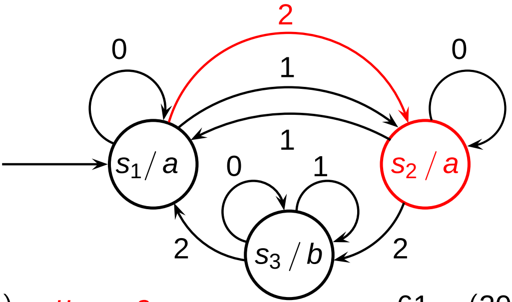

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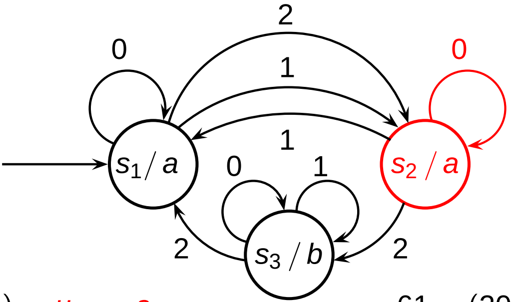

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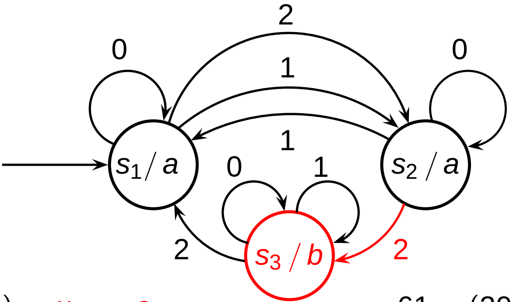

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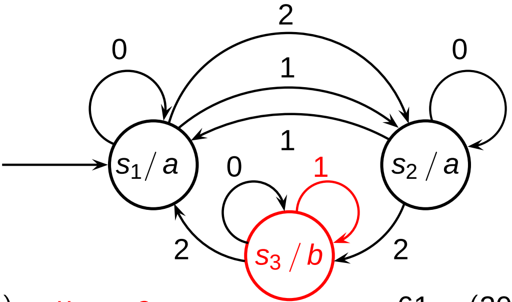

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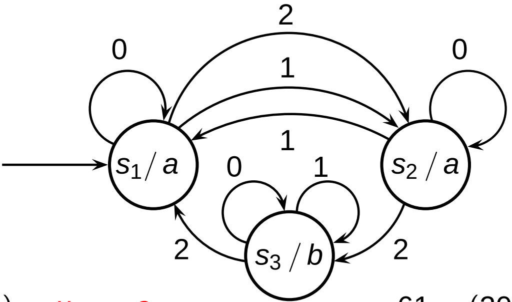

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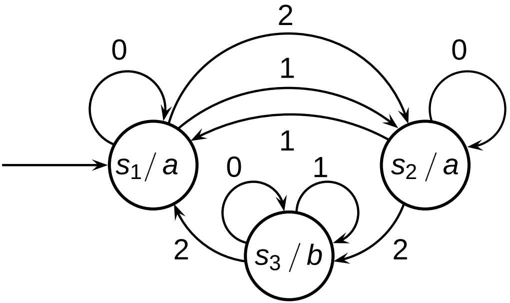
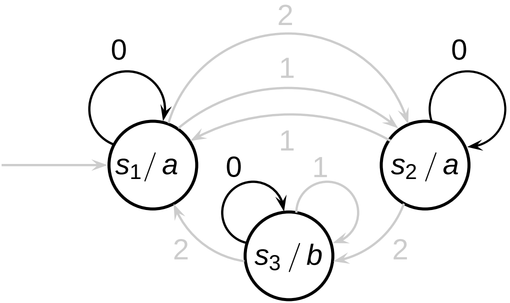
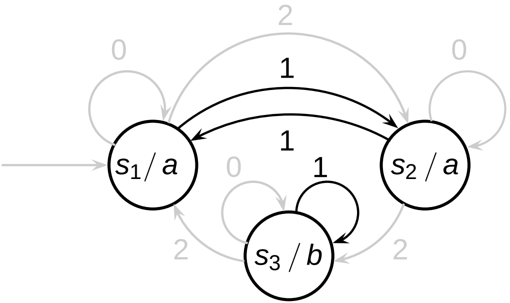
$(u_n)_{n\geqslant 0}$: *aaaaabaabaabaaabbaaabaaabbaaabaaabbaaaaaaba*...


2
0
1
0
1
0
1
s_1 / a
s_2 / a
2
2
s_3 / b



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



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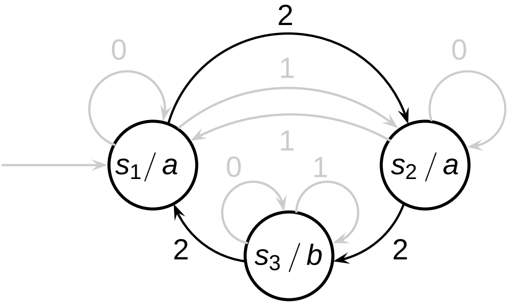

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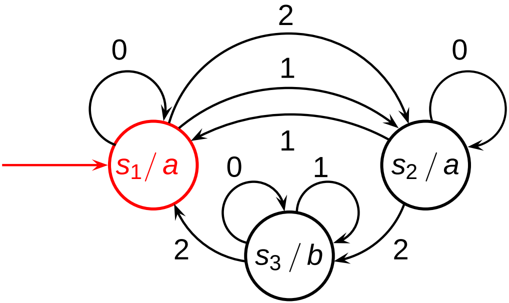

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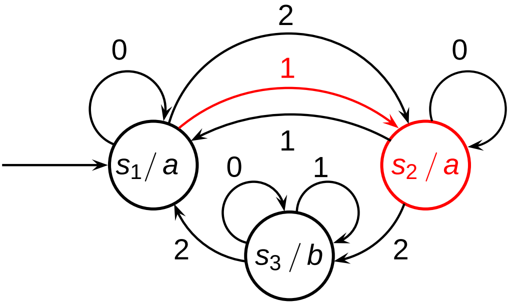

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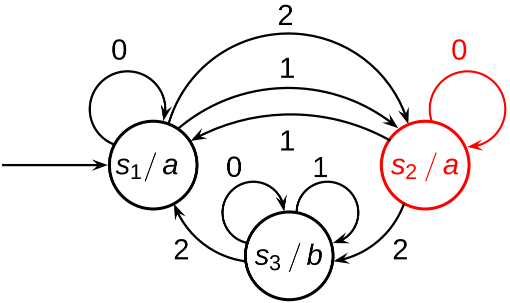
$$M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$



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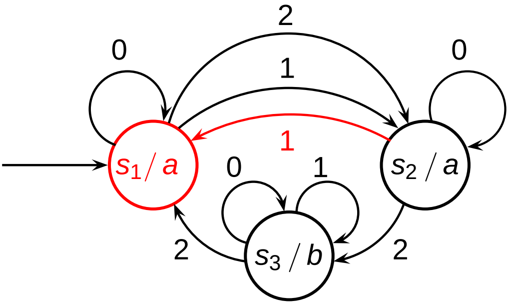

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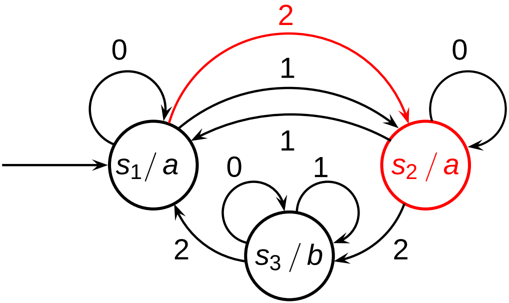

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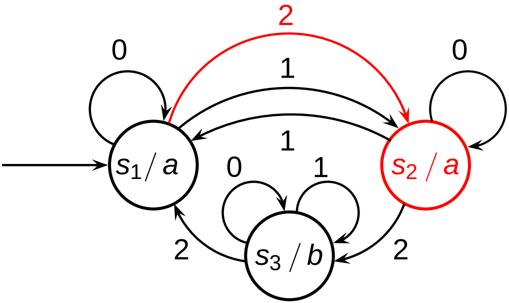

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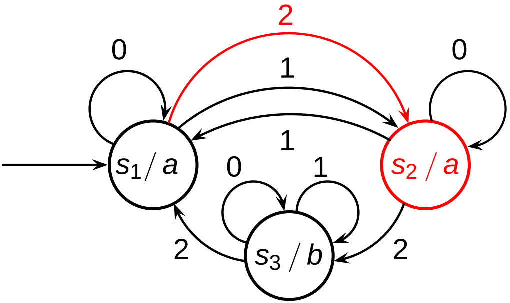
$$\boxed{S(n) := M_{\varepsilon_0(n)} M_{\varepsilon_1(n)} \cdots M_{\varepsilon_{\ell-1}(n)}}$$

$$\boxed{u_n = f(S(n)\mathbf{e}_1)} \qquad \mathbf{e}_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$$



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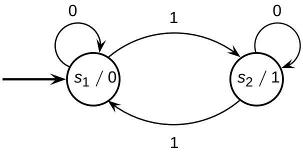
Definition

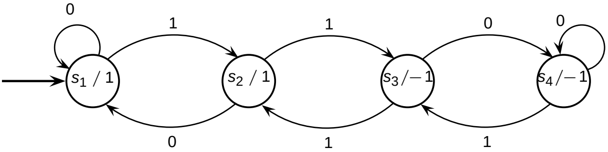
A q-automatic sequence is called *invertible* if there exists an automaton such that all transition matrices are invertible and M_0 is the identity matrix.

Examples of automatic sequences

Thue-Morse sequence (invertible):



Rudin-Shapiro sequence (**not** invertible):



★ Automatic Sequences and Generalized Thue-Morse sequences

- u_n ... invertible automatic sequence

$$\Longrightarrow \quad \boxed{u_n = f(S(n)\mathbf{e}_1)},$$

where $S(n)$ is a generalized Thue-Morse sequence on $H = \mathrm{SL}(m, \mathbb{R})$

★ Gelfond's 3rd problem on polynomial values

★ The sum-of-digits function:

Theorem (D., Mauduit and Rivat, 2011)

Let $d \geqslant 2$ be an integer, $\boxed{q \geqslant q_0(d)}$ a sufficently large **prime** *number, and $P \in \mathbb{N}[X]$ of degree d such the leading coefficient is co-prime to q. Set $r = \gcd(q-1, m)$. Then we have*

$$\boxed{\lim_{N\to\infty} \frac{1}{N} \#\{n < N : s_q(P(n)) \equiv a \bmod m\} = \frac{1}{r} Q(a,r)},$$

where $Q(a,r) = \#\{0 \leqslant n < r : P(n) \equiv a \bmod r\}$.
Furthermore, $(\alpha s_q(P(n)))_{n\in\mathbb{N}}$ is uniformly distributed modulo 1 iff α is irrational.

Remark. $q_0(d) \leqslant e^{67d^3(\log d)^2}$

★ Gelfond's 3rd problem on polynomial values

★ **Proof method:**

- Exponential sums ($e(x) := \exp(2\pi i x)$):

$$\sum_{n<N} e\left(\alpha s_q(P(n))\right)$$

- Van-der-Corput inequality
- Fourier terms

$$F_\lambda(\alpha, h) = \frac{1}{q^\lambda} \sum_{0 \leqslant u < q^\lambda} e\left(\alpha s_q(u) - huq^{-\lambda}\right)$$

- Estimates on Fourier terms and exponential sum estimates

★ Subsequences of the form $\lfloor n^c \rfloor$

★ The sum-of-digits function:

Theorem (Deshouillers, D. and Morgenbesser, 2012)

Let u_n be a q-automatic sequence (on an alphabet $\mathcal{A}$) and

$$1 < c < 7/5.$$

Then for each $\alpha \in \mathcal{A}$ then asymptotic density of α in the subsequence $u_{\lfloor n^c \rfloor}$ exists if and only if the asymptotic density of α in u_n exists (and they are the same).

In particular it follows that

$$\boxed{\lim_{x\to\infty} \frac{1}{N} \# \{n < N : s_q(\lfloor n^c \rfloor) \equiv a \bmod m\} = \frac{1}{m}}.$$

It is conjectured that this holds for all non-integers $c > 1$.

★ Gelfond's 2nd problem on primes

★ **The sum-of-digits function:**

Theorem (Mauduit and Rivat, 2010)

Let $q, m \geqslant 2$. *Then*

$$\boxed{\lim_{N\to\infty} \frac{1}{\pi(x; a, (m, q-1))} \# \{p < N : s_q(p) \equiv a \bmod m\} = \frac{(m, q-1)}{m}}.$$

Furthermore, $(\alpha s_q(p))_{p\in\mathbb{P}}$ *is uniformly distributed modulo* 1 *iff* α *is irrational.*

★ Local results the sum-of-digits function on primes

Theorem (D., Mauduit and Rivat, 2009)

Suppose that $(q, k-1) = 1$. *Then*

$$\#\{primes\ p < N : s_q(p) = k\}$$
$$= \frac{q-1}{\varphi(q-1)} \frac{\pi(N)}{\sqrt{2\pi\sigma_q^2 \log_q N}} \left(\exp\left(-\frac{(k-\mu_q \log_q N)^2}{2\sigma_q^2 \log_q N} \right) + O((\log N)^{-\frac{1}{2}+} \right.$$

where

$$\mu_q := \frac{q-1}{2}, \quad \sigma_q^2 := \frac{q^2-1}{12}.$$

Remark: This asymptotic expansion is only significant if

$$\left| k - \mu_q \log_q N \right| \leqslant C (\log N)^{\frac{1}{2}}$$

Note that $\frac{1}{\pi(N)} \sum_{p<N} s_q(p) \sim \mu_q \log_q N$.

★ Binary Representation of Primes

Corollary

Theorem (D., Mauduit, Rivat (2011))

$s_2(n)$ ... *number of powers of 2 in the binary expansion of n*

$$\#\{\text{primes } p < 2^{2k} : s_2(p) = k\} \sim \frac{2^{2k}}{\sqrt{2\pi}\log 2\, k^{\frac{3}{2}}}$$

★ Local results the sum-of-digits function on primes

Lemma

For every fixed integer $q \geq 2$ there exist two constants $c_1 > 0$, $c_2 > 0$ such that for every k with $(k, q-1) = 1$

$$\sum_{p \leqslant N\ p \equiv k \bmod q-1} e(\alpha s_q(p)) \ll (\log N)^3 N^{1-c_1\|(q-1)\alpha\|^2}$$

uniformly for real α with $\|(q-1)\alpha\| \geqslant c_2 (\log N)^{-\frac{1}{2}}$.

Remark. This is a refined version of the result by Mauduit and Rivat.

★ Local results the sum-of-digits function on primes

Lemma

Suppose that $0 < \nu < \frac{1}{2}$ *and* $0 < \eta < \frac{\nu}{2}$. *Then for every* k *with* $(k, q-1) = 1$ *we have*

$$\sum_{p \leqslant N,\ p \equiv k \bmod q-1} e(\alpha s_q(p)) = \frac{\pi(N)}{\varphi(q-1)} e(\alpha \mu_q \log_q N)$$

$$\times \left(e^{-2\pi^2\alpha^2\sigma_q^2 \log_q N} \left(1 + O\left(|\alpha|\right)\right) + O\left(|\alpha| (\log N)\right)\right.$$

uniformly for real α *with* $|\alpha| \leqslant (\log N)^{\eta - \frac{1}{2}}$.

Remark. This is a refined version of a **central limit theorem** and can be proved by a refined moment method.

Thank you!