Optimal Chebyshev Bounds for Beurling Generalized Numbers

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Abstract

0) Thanks to conference organizers and the Hungarian mathematical community.

1) A survey of Beurling g-numbers.

2) Weak conditions for Chebyshev O-bounds

$$\frac{x}{\log x} \ll \pi(x) \ll \frac{x}{\log x}$$

for g-numbers, where $\pi(x)$ is the counting function of g-primes.

3) The conditions for these bounds are best-possible.

g-Number Definitions

A g-prime system.

$\mathcal{P}: p_1 \le p_2 \le p_3 \ldots, \quad p_1 > 1, \quad p_i \to \infty.$

(The p_i are assumed to be real, not necessarily rational integers.)

A g-integer system. Semigroup generated by $\mathcal{P}$.

$\mathcal{N}$: $1 = n_0 < n_1 \le n_2 \le n_3 \le \ldots$

Call the p_i *g-primes* and the n_i *g-integers*.

Remarks. Unique prime factorization not assumed;
g-primes and g-integers may be repeated;
g-integers usually don't have additive structure.

Central problems of g-number theory: Show that if g-integers are distributed much like the rational integers, then g-primes must also be so distributed; and conversely.

Examples

Ex. $\mathcal{P}_1$: 3, 5, 7, 11, ...,

$\mathcal{N}_1$: Odd numbers (Density $= 1/2$)

Ex. $\mathcal{P}_2$: 3, 3, 5, 7, 11, 13, ...

$\mathcal{N}_2$: 1, 3, 3, 5, 7, 9, 9, 9, 11, 13, 15, 15, 17, ...

(Density $= 3/4$)

Ex. $\mathcal{P}_3$: 1.5, 5, 7, 11, 13, ...

$\mathcal{N}_3$: 1, 1.5, 2.25, 3.375, 5, 5.0625, 7, ...

(Density $= 1$)

Key Functions

$$\pi(x) = \pi_{\mathcal{P}}(x) := \#\{\mathcal{P} \cap [1, x]\}$$

$$N(x) = N_{\mathcal{P}}(x) := \#\{\mathcal{N} \cap [1, x]\}$$

$$\Pi(x) := \pi(x) + \tfrac{1}{2}\pi(x^{1/2}) + \tfrac{1}{3}\pi(x^{1/3}) + \dots$$

$$\zeta(s) := \textstyle\sum_{k \geq 0} {n_k}^{-s} = \prod_{k \geq 1} (1 - {p_k}^{-s})^{-1}, \quad \Re s > 1$$

$$\log \zeta(s) = \textstyle\int_1^\infty x^{-s}\, d\Pi(x), \quad \Re s > 1$$

$$-\zeta'(s)/\zeta(s) = \textstyle\int_1^\infty x^{-s} \log x \, d\Pi(x), \quad \Re s > 1$$

$$\psi(x) := \textstyle\int_1^x \log t \, d\Pi(t) = \sum_{{p_i}^{\alpha_i} \leq x} \log p_i$$

Beurling's PNT

Beurling's Theorem. Let $\mathcal{P}$ be a g-prime system satisfying

$$N(x) - Ax = O(x \log^{-\gamma} x) \tag{B}$$

for some $A > 0$ and $\gamma > 3/2$. Then

$$\pi(x) \sim x/\log x$$

for $x \to \infty$, i.e. the PNT holds for $\mathcal{P}$.

Beurling's result is "(B)-Optimal" in that there exists a g-prime system for which

$$N(x) - Ax = O(x \log^{-3/2} x)$$

and for which the PNT does not hold.

Chebyshev – Type Inequalities

We seek conditions on $N_{\mathcal{P}}(x)$ that ensure

$$x/\log x \ll \pi(x) \ll x/\log x. \tag{C}$$

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Chebyshev Bound Theorem (H. D.) If $\mathcal{P}$ is a g-prime system for which $N(x) - Ax = O(x \log^{-\gamma} x)$ for $\gamma > 1$, then (C) holds.

Ex. (R. S. Hall) (C) can fail for $\gamma < 1$. ($\therefore$ Th. is (B)–Optimal.)

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J.-P. Kahane: The (L^1) conjecture is not true:

He gave an example of (L^1) but both inequalities in (C) fail.

Conditions for Chebyshev Bounds with (L^1)

Th. (J. Vindas, 2012). Suppose $\mathcal{N}$ is a system satisfying

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and $N(x) - Ax = o(x/\log x)$. Then (C) holds.

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Th. (HD, 2012; JV, 2012). Suppose (L^1) holds for $\mathcal{N}$ and

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Then the upper Chebyshev bound holds. Also (HD, 2012), if the O-constant is small enough, then the lower (C) bound holds too.

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Th. (WBZ, 2012). Suppose $\mathcal{N}$ is a system satisfying (L^1) and

$$\int_1^x |N(u) - Au|\, u^{-1} \log\, u\, du \ll x.$$

Then the upper Chebyshev bound holds. The lower Chebyshev bound follows from an analogous integral condition.

Outline of Chebyshev Estimate I

The starting point is the identity for $\sigma = \Re s > 1$,

$$\int_1^\infty x^{-s}\,\frac{\psi(x) - x}{x}\,dx = \frac{-\zeta'}{s\,\zeta}(s) - \frac{1}{s-1} =: g(s).$$

An average value of $\{\psi(y) - y\}/y$ (as $y \to \infty$) is represented, using the Wiener-Ikehara method, as

$$\lim_{\epsilon \to 0+} \frac{1}{2} \int_{-2\lambda}^{2\lambda} \Big(1 - \frac{|t|}{2\lambda}\Big) e^{ity} g(1 + \epsilon + it)\,dt + o_\lambda(1).$$

$\zeta(s)$ has pole-like behavior at $s = 1$. This insures that $\zeta(1 + it) \neq 0$ if $|t|$ is small enough. So we use a small λ.

($\therefore$ no asymptotic for $\psi(x)/x$.)

Main task: show that $g(1 + \epsilon + it)$ is tractable for small $|t|$ as $\epsilon \to 0+$.

Outline of Chebyshev Estimate II

For $\sigma \geq 1$, write

$$f(s) := \frac{1}{A} \int_1^\infty x^{-s-1}(N(x) - Ax)\, dx.$$

This is bounded by hypothesis. From last slide,

$$g(s) = \frac{-\zeta'}{s\,\zeta}(s) - \frac{1}{s-1} := F_1(s) + F_2(s) + F_3(s),$$

with $F_1(s)$, $F_2(s)$ easy to handle and

$$F_3(s) = \frac{itf'(s)}{(1+it)\{1+itf(s)\}} + o(1), \quad \sigma \to 1+\,.$$

Outline of Chebyshev Estimate III

Main term in average of $\{\psi(y) - y\}/y$ is

$$\lim_{\epsilon \to 0+} \frac{1}{2} \int_{-2\lambda}^{2\lambda} \Big(1 - \frac{|t|}{2\lambda}\Big) e^{ity} \frac{it\, f'(1+\epsilon+it)}{(1+it)\{1+itf(1+\epsilon+it)\}}\, dt.$$

Use (B1) to control contribution of f', and write $1/\{1+itf(s)\}$ as an absolutely convergent series for $|t|$ small. Then estimate the resulting convolution integrals.

(This is the idea that underlies Wiener's Division Theorem.)

For powers of t, estimate derivatives of Fejér kernel. Technical!

Optimality of Conditions (L^1) and $(B1)$ for (C)

$$\int_1^\infty |N(x) - Ax|\, x^{-2}\, dx < \infty \qquad (L^1)$$

$$N(x) - Ax = O(x/\log x). \qquad (B1)$$

Th. (HD-WBZ, 2012). Given any positive, increasing unbounded function $f(x)$ on $[1, \infty)$ (no matter how slowly growing), there exists a g-number system $\mathcal{N}_B$ such that

(1) The counting function $N_B(x)$ of the g-integers satisfies (L^1) and

$$N_B(x) - Ax = O(f(x)x/\log x). \qquad \text{(Bf)}$$

(2) The associated zeta function $\zeta_B(s)$ is analytic on the open half plane $\{s : \sigma > 1\}$. Also, $(s-1)\,\zeta_B(s)$ has a continuous extension to the closed half plane $\{\sigma \geq 1\}$ and $it\zeta_B(1+it) \neq 0$;

(3) Each of the Chebyshev bounds (C) fails.

Outline of $\mathcal{P}_B$ Construction

Three preliminaries.

1. Form another unbounded function $k(x)$ on $[1, \infty)$ that is
   - even more slowly growing than $f(x)$,
   - continuously differentiable,
   - satisfies $(\log x)/k(x) \uparrow$.
2. Form a very rapidly increasing sequence (A_n) for which

$$\sum_{n \geq 1} \frac{\log k(n)}{k(A_n)} < \infty.$$

3. Define $A_n^{\star} = A_n \sqrt{k(n)}$.

Recipe for $\mathcal{P}_B$

1. Use rational primes on $[2, A_{n_0})$, for sufficiently large n_0.

2. A_{n_0}: a g-prime with multiplicity $[\{A_{n_0} \log k(n_0)\}/\{2 \log A_{n_0}\}]$.

3. On $(A_{n_0}, A^\star_{n_0}]$, no primes of $\mathcal{P}_B$.

4. On $(A^\star_{n_0}, A_{n_0+1})$, use the rational primes in this interval.

5. A_{n_0+1} and after: repeat the same pattern.

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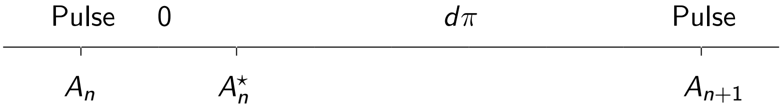

Figure 1. $d\pi_B$ on one interval

Failure of Chebyshev Bounds

1. $\pi_B(x)$ is too large on the sequence (A_n):

$$\frac{\pi_B(A_n)}{A_n/\log A_n} \geq \frac{[A_n \log k(n)/(2\log A_n)]}{A_n/\log A_n} \sim \frac{\log k(n)}{2} \to \infty.$$

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2. $\pi_B(x)$ is too small on the sequence $(A_n^\star) := (A_n\sqrt{k(n)})$:

$$\frac{\pi_B(A_n^\star)}{A_n^\star/\log(A_n^\star)} \leq \frac{\pi(A_n) + A_n \log k(n)/(2\log A_n)}{A_n^\star/\log(A_n^\star)} \ll \frac{\log k(n)}{k(n)^{1/2}} \to 0.$$

Still to Do

It remains to show $\mathcal{N}_B$ satisfies the conditions

$$\int_1^\infty |N(x) - Ax|\, x^{-2}\, dx < \infty \qquad (L^1)$$

$$N(x) - Ax = O(x\, f(x)/\log x). \qquad (Bf)$$

Exponential Representation of dN

For any g-number system, dN and $d\Pi$ are related by

$$dN = \delta_1 + d\Pi + \frac{1}{2!} d\Pi * d\Pi + \frac{1}{3!} d\Pi * d\Pi * d\Pi + \cdots =: \exp d\Pi$$

where δ_1 = Dirac measure at 1, * = multiplicative convolution, and convergence is uniform on compact sets. (This formula is valid also for rational primes and integers.)

The exp function on measures does what we expect of an exponential, with $*$ as the multiplication:

$$\exp(d\alpha + d\beta) = \exp(d\alpha) * \exp(d\beta).$$

Formula for $N_B(x)$

Using exponentials, write

$$dN_B = \exp d\Pi_B = (\exp d\Pi) * (\exp\{d\Pi_B - d\Pi\}).$$

Here $d\Pi$ is the weighted counting measure of *rational* primes, and dN is the counting measure of rational integers.

$$\begin{aligned} N_B(x) &= \iint_{st \le x} dN(s)\,(\exp\{d\Pi_B - d\Pi\})(t) \\ &= \int_{t \le x} \int_{s \le x/t} \ldots = \int_{1-}^{x} \left[\frac{x}{t}\right] \exp d(\Pi_B - \Pi)(t) \\ &= \int_{1-}^{x} \left(\frac{x}{t} + O(1)\right) \exp d(\Pi_B - \Pi)(t). \end{aligned}$$

Estimates for $N_B(x)$

$$N_B(x) = \int_{1-}^{x} \frac{x}{t} \exp d(\Pi_B - \Pi)(t) + O(1) \int_{1-}^{x} \exp d\Pi_0(t),$$

where $d\Pi_0$ is the variation measure of $d\Pi_B - d\Pi$.

The last term is $\ll N_0(x) \ll x\, k(x)/\log x$, where $N_0(x)$ is the g-number counting function generated by Π_0.

The main term in $N_B(x)$ formula is $M(x) - E(x)$, where

$$M(x) = x \int_{1-}^{\infty} t^{-1} \exp d(\Pi_B - \Pi)(t) = A\, x$$

and

$$E(x) = x \int_{x}^{\infty} t^{-1} \exp d(\Pi_B - \Pi)(t).$$

To do: Show $A < \infty$, and bound $N_0(x)$ and $E(x)$.

Key Inequality I

$$\int_1^\infty x^{-1} \frac{\log x}{k(x)}\, d\pi_0(x) < \infty, \qquad (\star)$$

where $d\pi_0$ is the total variation measure of $d\pi_B - d\pi$.

Thus, the counting measure of $\mathcal{P}_B$ and that of the rational primes are so close that the Dirichlet series of $\log\{\zeta_B(s)/\zeta(s)\}$ is absolutely convergent on the closed half-plane $\{s : \sigma \geq 1\}$.

(Note that if $(\star)$ held with a *bounded* function $k(x)$, then $(d/ds)\log\{\zeta_B(s)/\zeta(s)\}$ also would be an absolutely convergent Mellin integral, and life would be much simpler.)

An Application of Key Inequality I

Since $d\pi_0$ is the biggest contributor to $d\Pi_0$, we also have

$$\int_1^\infty x^{-1}\,\frac{\log x}{k(x)}\,d\Pi_0(x) < \infty.$$

The counting function of $\mathcal{N}_0$, the g-number system generated by $d\pi_0$, is $dN_0 = \exp d\Pi_0$.

An easy calculation then gives

$$\int_1^\infty x^{-1}\,\frac{\log x}{k(x)}\,dN_0(x) < \infty.$$

Key Inequality II

For large x,

$$\Big| \int_x^{\infty} t^{-1} d(\pi_B - \pi)(t) \Big| \leq \begin{cases} \log k(n) / \log A_n, & \text{if } A_n < x \leq A_n^{\star}; \\ \frac{1}{4}(\log k(n+1)/\log A_{n+1})^2, & \text{if } A_n^{\star} < x \leq A_{n+1}. \end{cases}$$

Remark. In Key Inequality I, $d\pi_0$ is total variation measure; here we have (+) contributions at "pulse points" and (-) contributions on intervals where $d\pi_B = 0$.
The integral inequality asserts that they largely cancel out.

Using the key relations, we show that

(1) N_B satisfies (Bf)

(2) ζ_B is well behaved for $\Re s \geq 1$. $\square$

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