On the KŁR conjecture in random graphs

D. Conlon, W. T. Gowers, W. Samotij and M. Schacht

July 5, 2013

Regularity

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ϵ-regular pair

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Often the strength of the regularity lemma lies in the fact that it can be combined with an appropriate counting lemma.

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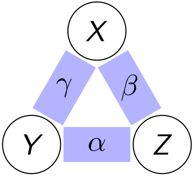

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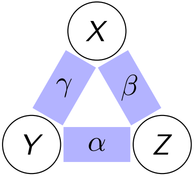

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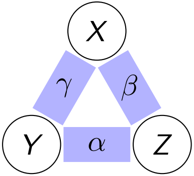

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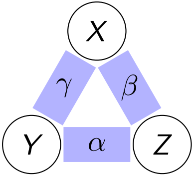

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Therefore, all but $2\epsilon|X|$ vertices in X have $|N_Y(x)| \geq (\gamma - \epsilon)|Y|$ and $|N_Z(x)| \geq (\beta - \epsilon)|Z|$.

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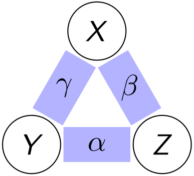

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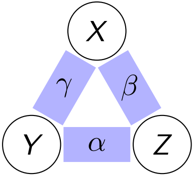

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Therefore, the number of triangles is at least

$$(1 - 2\epsilon)|X|(\gamma - \epsilon)|Y|(\beta - \epsilon)|Z|(\alpha - \epsilon) \approx \alpha\beta\gamma|X||Y||Z|.$$

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$$n^k \left(\prod_{ij \in E(H)} d_{ij} \pm \delta \right).$$

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For every $\epsilon, D > 0$, there exist $\eta > 0$ and T such that every graph G which is (η, p, D)-upper-uniform has an (ϵ, p)-regular partition $V_1, V_2, \ldots, V_t$ into $t \leq T$ pieces.

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The upper-uniformity condition was recently removed by Scott.

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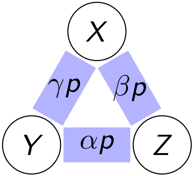

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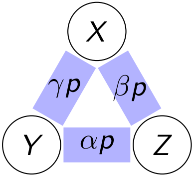

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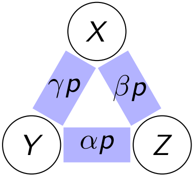

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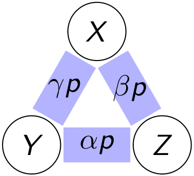

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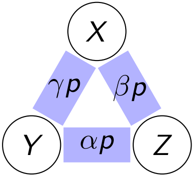

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But now we can say nothing about the density of edges between $N_Y(x)$ and $N_Z(x)$!

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The resulting graph G' will still be such that the graph between each pair of vertex sets is (ϵ, p)-regular but it contains no triangles.

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For $p \gg n^{-1/2}$, there are very few graphs on vertex set $X \cup Y \cup Z$ with $|X| = |Y| = |Z| = n$ such that the graph between each pair of vertex sets is (ϵ, p)-regular and the graph contains no triangles.

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So few that the random graph $G_{N,p}$ with $N = O(n)$ is unlikely to contain any such bad example as a subgraph.

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For every graph H on vertex set $\{1, 2, \ldots, k\}$ and $p \gg n^{-1/m_2(H)}$, there are very few graphs on vertex set $V_1 \cup V_2 \cup \cdots \cup V_k$ with $|V_1| = |V_2| = \cdots = |V_k| = n$ such that the graph between each pair of vertex sets V_i and V_j with $ij \in E(H)$ is (ϵ, p)-regular and the graph contains no copies of H.

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Here

$$m_2(H) = \max \left\{ \frac{e(H') - 1}{v(H') - 2} : H' \subseteq H, v(H') \geq 3 \right\}$$

and $p = n^{-1/m_2(H)}$ is roughly where every edge of the random graph $G_{n,p}$ is contained in a copy of H.

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All graphs - Saxton and Thomason

Counting

The KŁR conjecture may be used to show that with high probability $G_{N,p}$ has the property that any subgraph defined on a large subset $V_1 \cup V_2 \cup \cdots \cup V_k$ and such that (V_i, V_j) is (ϵ, p)-regular for all $ij \in E(H)$ contains a single copy of H.

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What if instead one wishes to know that every such subgraph contains many copies of H?

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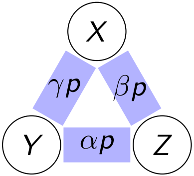

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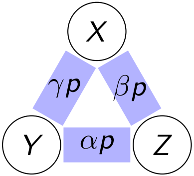

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it may be used to show that in any subgraph of $G_{n,p}$ consisting of three large vertex sets X, Y and Z with an (ϵ, p)-regular graph between each pair of vertex sets, there are approximately $\alpha\beta\gamma p^3|X||Y||Z|$ triangles.

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Proofs use two different methods, developed independently by C.-Gowers and by Schacht for proving combinatorial theorems relative to random sets.

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- More...

Thank you for your attention!