FORWARD BROWNIAN MOTION

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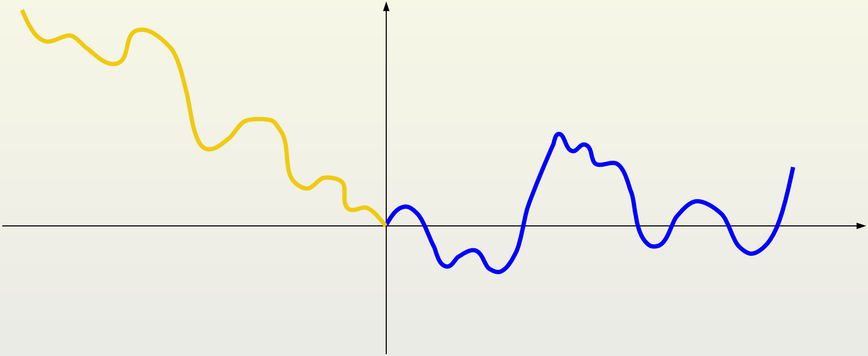
Collaborator and preprint

Joint work with Michael Scheutzow.

Math Arxiv:
http://arxiv.org/abs/1302.6958

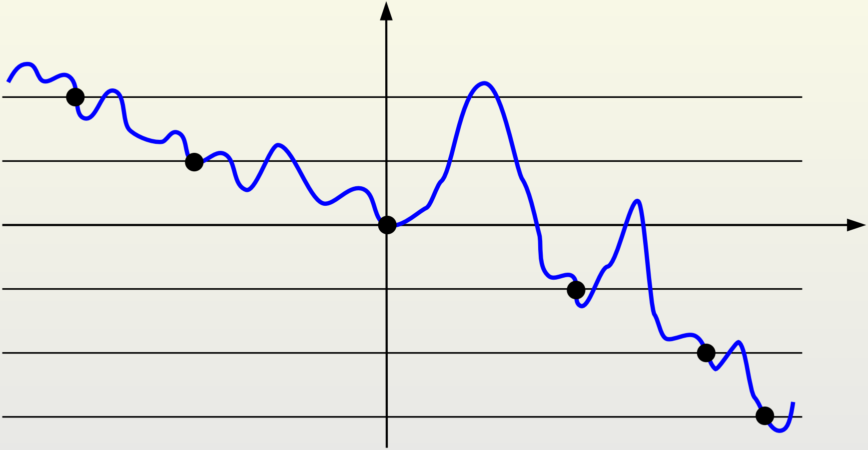
Example

Concatenation of time-reversal of 3-dimensional Bessel process and independent Brownian motion.



A different look at the same example

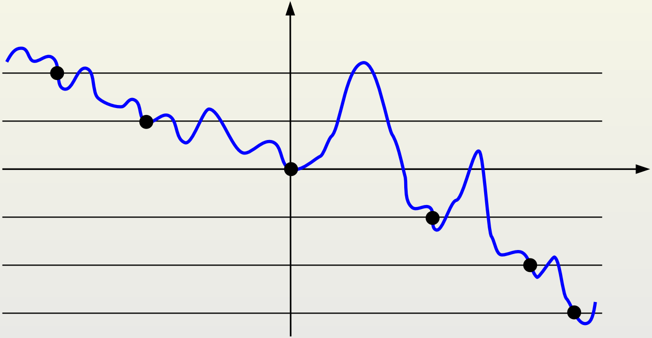
Concatenation of independent pieces of Brownian motion.
(D. Williams (1974))



Forward Brownian motion

DEFINITION

We say that X is forward Brownian motion (FBM) if there exist random times S_k, $k \in \mathbb{Z}$, such that $\lim_{k\to-\infty} S_k = -\infty$ and for every k, $\{X(S_k + t) - X(S_k), t \geq 0\}$ is Brownian motion.



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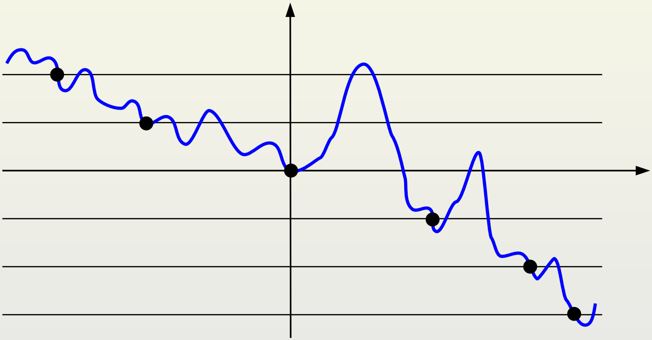

Related models: Bertoin & Savov (2011), Kemeny, Snell & Knapp (1976)

Decomposable FBM

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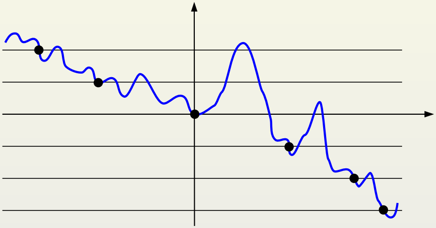

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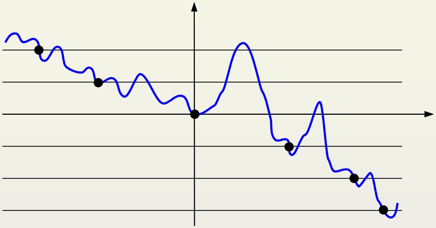
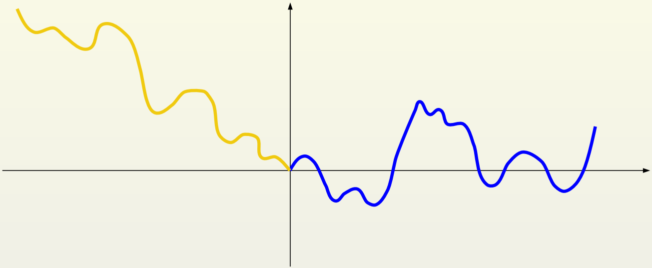


QUESTION

Is every FBM decomposable?

Path properties of FBM's



3-dimensional Bessel process goes to infinity at the rate $\sqrt{t}$ (up to logarithmic factors on lower and upper side; Shiga and Watanabe (1973)).

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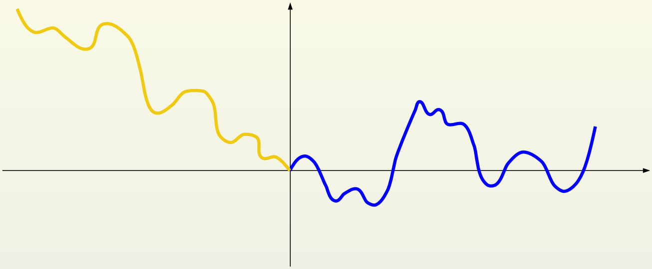

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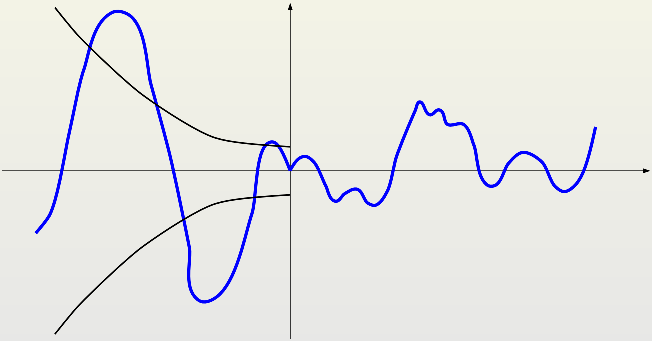
Can FBM go to infinity faster than that?

Maximal rate of growth

THEOREM

For each increasing function $f : [0, \infty) \to [0, \infty)$ there exists a strongly decomposable FBM X for which, a.s.,

$$\limsup_{t \to -\infty} (X_t - f(-t)) \geq 0 \qquad \text{and} \qquad \liminf_{t \to -\infty} (X_t + f(-t)) \leq 0.$$



An eigenvalue problem

Let

$$\mathcal{A} = \frac{1}{2}\left(\frac{d^2}{dx^2} - x\frac{d}{dx}\right),$$

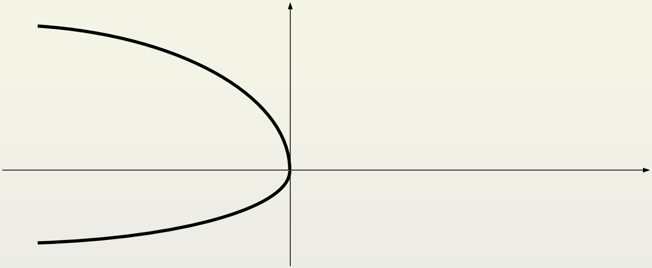
and $m(dx) = 2e^{-x^2/2}dx$. For all $\boxed{-\infty \le c_1 < c_2 \le \infty}$ there is a complete orthonormal system in $L^2([c_1, c_2], m)$ of eigenfunctions of the Sturm-Liouville problem

$$\mathcal{A}\psi = \lambda\psi, \qquad \psi(c_i) = 0, i = 1, 2, \qquad \text{if } |c_i| < \infty,$$

whose corresponding eigenvalues are simple and non-positive. Let $\boxed{-\lambda_0(c_1, c_2)}$ denote the largest eigenvalue.

Minimum asymptotic range

$-\infty \le c_1 < c_2 \le \infty$
$\mathcal{R} := \{(t,x) : t < 0, c_1\sqrt{|t|} < x < c_2\sqrt{|t|}\}$



Minimum asymptotic range (2)

$-\infty \le c_1 < c_2 \le \infty$

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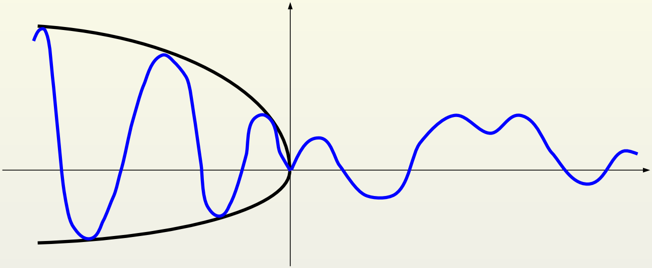


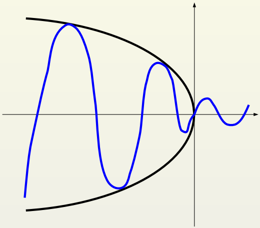
THEOREM

(i) If $\lambda_0(c_1,c_2) < 1$ then there exists an FBM with trajectories that fit into $\mathcal{R}$ asymptotically, as $t \to -\infty$.

(ii) If $\lambda_0(c_1,c_2) > 1$ then there does not exist an FBM with trajectories that fit into $\mathcal{R}$ asymptotically, as $t \to -\infty$.

Critical shapes

$\lambda_0(c_1, c_2) = 1$ for the following pairs (c_1, c_2).



$c_1 = -1, c_2 = 1$

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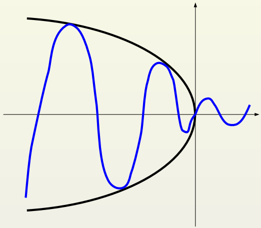

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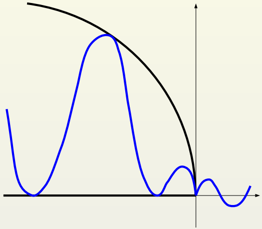

$c_1 = 0, c_2 \approx 2.12411$

Critical shapes

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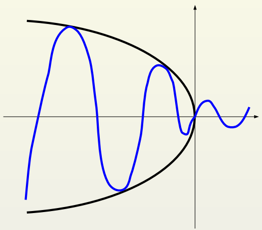

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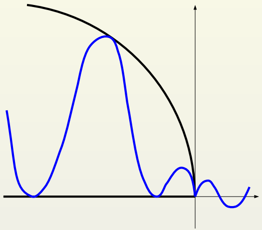

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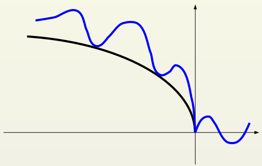

$c_1 = 1, c_2 = \infty$

Critical shapes

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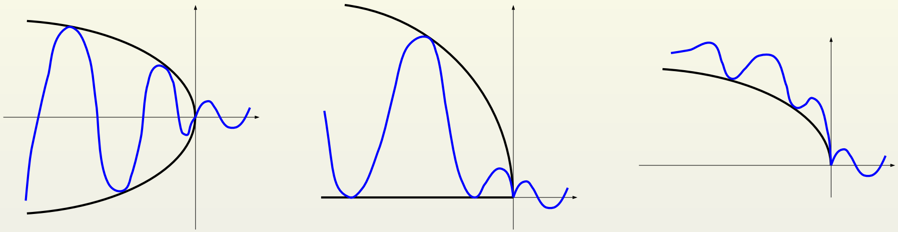


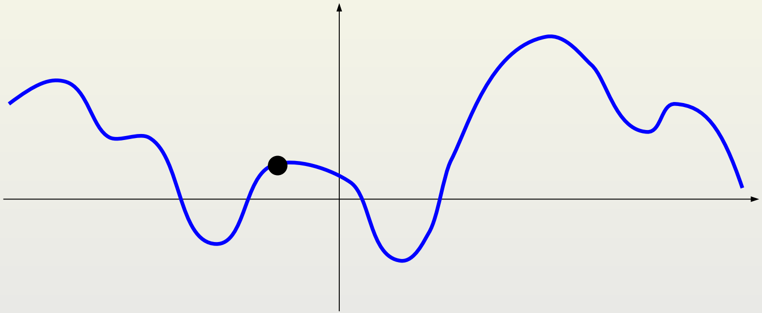
$c_1 = -1, c_2 = 1$ $\quad$ $c_1 = 0, c_2 \approx 2.12411$ $\quad$ $c_1 = 1, c_2 = \infty$

These critical shapes are the same as in the results on "slow points." Davis (1983), Greenwood and Perkins (1983), Perkins (1983).

Generic Brownian motion on the real line

DEFINITION

We will call $\{X_t,\ t \in \mathbb{R}\}$ *two-sided Brownian motion* (2BM) if there exists a random time S such that $\{X_{S+t} - X_S,\ t \geq 0\}$ and $\{X_{S-t} - X_S,\ t \geq 0\}$ are independent standard Brownian motions.



Which decomposable FBM's are 2BM's?

We say that FBM X is decomposable if it is the concatenation of independent pieces B^k of Brownian paths truncated at stopping times T_k. If (B^k, T_k) are i.i.d. than we call X strongly decomposable.

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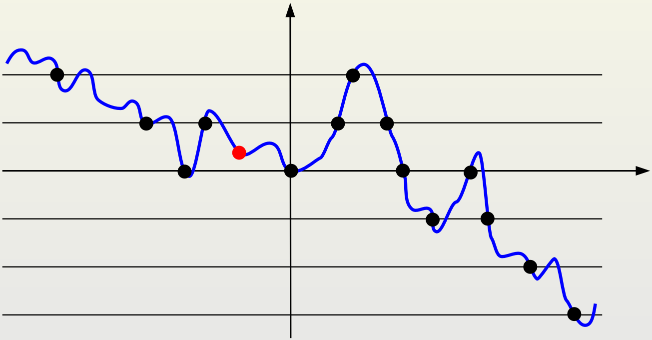
THEOREM

If X is strongly decomposable and $\mathbb{E} T_k < \infty$ then X is 2BM.



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(i) For every $p < 1$ there exists a strongly decomposable X with $\mathbb{E}T_k^p < \infty$ which is not 2BM.

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(i) For every $p < 1$ there exists a strongly decomposable X with $\mathbb{E} T_k^p < \infty$ which is not 2BM.
(ii) For any $p \in (0, \infty)$, there exists a decomposable FBM X satisfying $\sup_k \mathbb{E} T_k^p < \infty$ which is not a 2BM.

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OPEN PROBLEM

Is true that if X is decomposable and $\sup_k |T_k| \leq 1$, a.s., then X is a 2BM?

FBM and BBM

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THEOREM

No.

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There is an FBM X that is not decomposable.

The above theorem is a corollary of the following result.

THEOREM

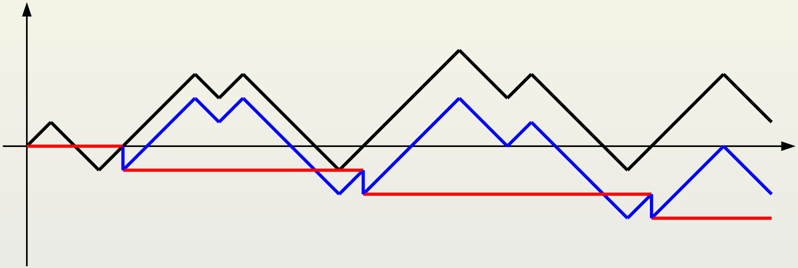
There exists an FBM X such that there is no random time T such that $\{X_{T+t} - X_T, t \geq 0\}$ and $\{X_{T-t} - X_T, t \geq 0\}$ are independent and $\{X_{T+t} - X_T, t \geq 0\}$ is standard Brownian motion.

Skew Brownian motion

Given a standard Brownian motion B and $-1 \le \beta \le 1$, the equation

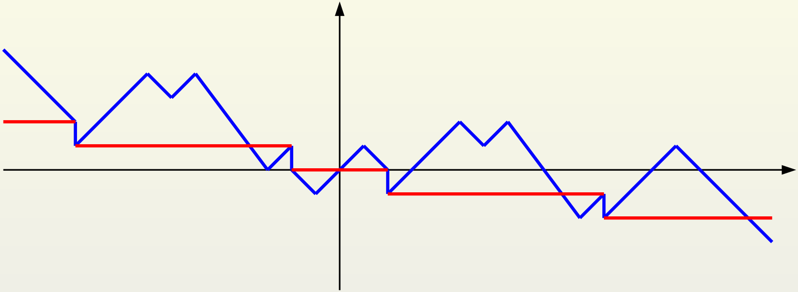
$$Z_t = B_t + \beta L_t^Z, \quad t \ge 0,$$

has a unique strong solution. Here L^Z is the symmetric local time of Z at 0. The process Z is called skew Brownian motion.



M. Barlow, H. Kaspi and A. Mandelbaum

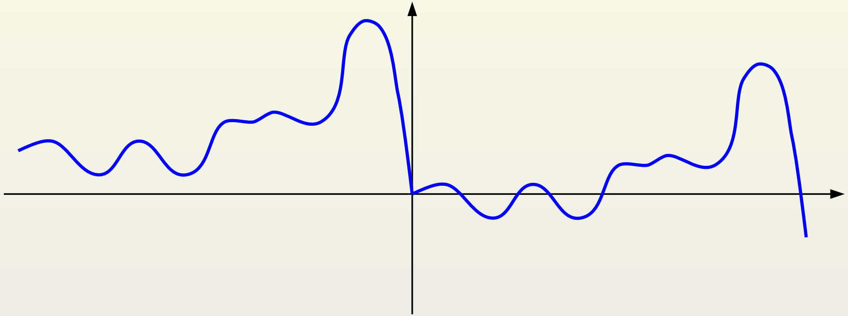
Skew Brownian motion-based FBM



$Z_t = B_t + \beta L_t^Z$

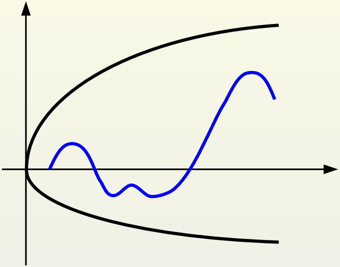
For $t \geq 0$, $\mathbb{E} X_{-t} = 2\beta\sqrt{2t/\pi}$.

FBM with big oscillations



Hitting of moving boundaries

$-\infty \le c_1 < c_2 \le \infty, \quad \mathcal{R} = \{(t,x) : t < 0, c_1\sqrt{|t|} < x < c_2\sqrt{|t|}\}$



B - Brownian motion, $B_1 = 0$, $T = \inf\{t \ge 1 : (t, B_t) \notin \mathcal{R}\}$

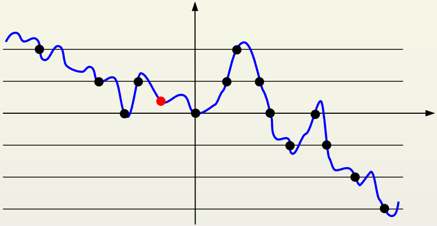
$$P(T \ge t) \approx t^{-\lambda_0(c_1,c_2)}$$

Breiman (1965), Novikov (1981), Uchiyama (1980)
Davis (1983), Greenwood and Perkins (1983), Perkins (1983)

Stationary point process and its Palm version

THEOREM

If X is strongly decomposable and $\mathbb{E} T_k < \infty$ then X is 2BM.



Let Λ be an ergodic simple point process on $\mathbb{R}$ and let Λ^* be its Palm version. One can find a random x such that translating Λ^* by x yields a configuration whose law is that of Λ.

Thorisson (1996); Slivnyak (1962), Zähle (1980); Kallenberg (2002)